Projective Differential Geometry of One-Parameter Families of Space Curves, and Conjugate Nets on a Curved Surface.

Second Memoir.*

BY GABRIEL M. GREEN.

§ 1. *Introduction and Fundamental Equations.*

In a preceding paper,† we have laid the foundations for a purely projective theory of conjugate nets on a surface and for the related theories of a one-parameter family of space curves and the congruence of tangents thereto. In fact, conjugate nets seemed to appear only incidentally, as a convenient avenue of approach to the apparently more general theory of the congruence of tangents to a one-parameter family of space curves. To make this point clearer: the equations

$$y^{(k)}=f^{(k)}(u, v), \qquad (k=1, 2, 3, 4) \tag{1}$$

represent for $v=$const. a one-parameter family of curves C_u, if the four y's be interpreted as homogeneous coordinates of a point in space. It is assumed that the surface defined by equations (1) is not developable, and also that the two one-parameter families of curves C_u ($v=$ const.) and C_v ($u=$const.) are not asymptotics on this surface. Under these conditions the four functions $y^{(k)}$ form a fundamental system of solutions of the completely integrable system of partial differential equations

$$\left.\begin{aligned} y_{uu}&=ay_{vv}+by_u+cy_v+dy, \\ y_{uv}&=a'y_{vv}+b'y_u+c'y_v+d'y, \end{aligned}\right\} \tag{2}$$

where $a'^2-a\neq 0$ and $a\neq 0$.‡ If $a'\neq 0$, the parameter net on the integral surface is not conjugate. Suppose $a'\neq 0$; then, in this, the general case, the family of curves conjugate to the family C_u is determined by making the transformation $\bar{u}=U(u,v)$, where $U(u,v)$ satisfies the partial differential equation of the first order, §

$$a'U_u-aU_v=0. \tag{3}$$

---

* The first three sections were presented to the American Mathematical Society February 28, 1914, and the last four October 30, 1915.

† G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII (1915), pp. 215–246. This will be cited as "preceding paper."

‡ Preceding paper, § 1.

§ Preceding paper, p. 221.

The new family of curves $\bar{u}=$const. will then be conjugate to the given family $v=$const., which is undisturbed by the transformation. In fact, the new net is defined by a fundamental system of solutions of a completely integrable system of partial differential equations of the form*

$$\left.\begin{aligned} y_{uu}&=ay_{vv}+by_u+cy_v+dy,\\ y_{uv}&=b'y_u+c'y_v+d'y. \end{aligned}\right\} \tag{4}$$

We have shown, in the memoir cited, that there is no loss of generality whatever in studying a one-parameter family of curves, defined by solutions of the differential equations (2), through the conjugate parameter net defined by system (4), in spite of the fact that the passage from equations (2) to equations (4) requires the integration—in general impossible—of the partial differential equation (3). The second of equations (4) is perfectly symmetric in the parameters u and v, but the first is not, and can not easily be made so without sacrifice of simplicity. It appears, then, as we have already said, that the differential equations (4) are more suited—if only on æsthetic grounds—for the study of the congruence of tangents to a one-parameter family of curves, than for that of the conjugate net determined thereby.

It was with these considerations in mind that in the preceding paper we treated the two component families of the conjugate net in an unsymmetric way. In particular, we chose a fundamental system of covariant points as follows. Three of them were a point on the surface and its corresponding first and minus first Laplace transforms; the fourth was the second Laplace transform. We shall in the present paper provide a substitute for this fourth point which will serve equally well in a theory of a one-parameter family of curves and in the theory of a conjugate net.

The introduction of this new covariant point in § 2 follows from a consideration of an important congruence of lines, which, with other congruences associated with the conjugate net, is studied in some detail in the course of the paper. A canonical development in non-homogeneous coordinates of the surface referred to the conjugate net is given, and leads to the consideration of a certain unodal cubic surface. Many questions of interest in the general theory of congruences are discussed in the latter half of the memoir. A conjugate net which is uniquely determined by the given one is introduced; we have ventured to call it the *associate conjugate net*, and lay the foundations for a systematic study thereof in its relation to the given conjugate net. In the last section is given a short treatment of isothermally conjugate nets, in the course of which a purely geometric characterization of these nets is obtained.

* Preceding paper, equations (16).

The completely integrable system (4), upon which we base our theory, has its coefficients restricted by certain integrability conditions. By differentiation, we may obtain from equations (4) the expressions *

$$\left.\begin{aligned}
y_{uuu}&=\alpha^{(1)}y_{vv}+\beta^{(1)}y_u+\gamma^{(1)}y_v+\delta^{(1)}y,\\
y_{uuv}&=\alpha^{(2)}y_{vv}+\beta^{(2)}y_u+\gamma^{(2)}y_v+\delta^{(2)}y,\\
y_{uvv}&=\alpha^{(3)}y_{vv}+\beta^{(3)}y_u+\gamma^{(3)}y_v+\delta^{(3)}y,\\
y_{vvv}&=\alpha^{(4)}y_{vv}+\beta^{(4)}y_u+\gamma^{(4)}y_v+\delta^{(4)}y,
\end{aligned}\right\}\tag{5}$$

where †

$$\left.\begin{aligned}
&\alpha^{(1)}=a(b+c')+a_u,\quad \beta^{(1)}=b_u+ab'_v+d+b^2+b'(c+ab'),\\
&\qquad\gamma^{(1)}=c_u+ac'_v+ad'+bc+c'(c+ab'),\\
&\qquad\delta^{(1)}=d_u+ad'_v+bd+d'(c+ab'),\\
&\alpha^{(2)}=ab',\quad \beta^{(2)}=b'(b+c')+b'_u+d',\quad \gamma^{(2)}=b'c+c'^2+c'_u,\\
&\qquad\delta^{(2)}=b'd+c'd'+d'_u,\\
&\alpha^{(3)}=c',\quad \beta^{(3)}=b'^2+b'_v,\quad \gamma^{(3)}=b'c'+c'_v+d',\quad \delta^{(3)}=b'd'+d'_v,\\
&\alpha^{(4)}=\frac{1}{a}(ab'-c-a_v),\quad \beta^{(4)}=\frac{1}{a}(b'c'+b'_u-b_v+d'),\\
&\gamma^{(4)}=\frac{1}{a}[b'c+c'(c'-b)+c'_u-c_v-d],\\
&\delta^{(4)}=\frac{1}{a}[b'd+d'(c'-b)+d'_u-d'_v].
\end{aligned}\right\}\tag{6}$$

It is tacitly assumed, of course, that $a\neq 0$. The conditions of complete integrability are ‡

$$\left.\begin{aligned}
\gamma^{(3)}+\alpha^{(3)}_v&=a\beta^{(4)}+\alpha^{(4)}_u,\\
(\alpha^{(3)}-b)\beta^{(4)}+b'\beta^{(3)}+\beta^{(3)}_v&=\alpha^{(4)}\beta^{(3)}+b'\gamma^{(4)}+\beta^{(4)}_u+\delta^{(4)},\\
c'\beta^{(3)}+\gamma^{(3)}_v+\delta^{(3)}&=\alpha^{(4)}\gamma^{(3)}+c\beta^{(4)}+\gamma^{(4)}_u,\\
\alpha^{(3)}\delta^{(4)}+d'\beta^{(3)}+\delta^{(3)}_v&=\alpha^{(4)}\delta^{(3)}+d\beta^{(4)}+d'\gamma^{(4)}+\delta^{(4)}_u,
\end{aligned}\right\}\tag{7}$$

the first of which, in virtue of equations (6), being equivalent to the equation

$$\frac{\partial}{\partial v}(b+2c')=\frac{\partial}{\partial u}\left(\frac{2ab'-c-a_v}{a}\right),\tag{8a}$$

or to

$$p_u=b+2c',\quad p_v=\frac{2ab'-c-a_v}{a}.\tag{8b}$$

§ 2. *The Axis Congruence.*

In the preceding paper, we discussed at some length the Laplace transforms of a conjugate net defined by a fundamental system of solutions of equations

* Preceding paper, § 3, equations (20). † Preceding paper, equations (21).
‡ Preceding paper, equations (23).

(4). We found for the minus first and first Laplace transforms the respective expressions

$$\rho = y_u - c'y, \qquad \sigma = y_v - b'y. \tag{9}$$

These two points lie with y in the tangent plane to the surface S_y. We wish to obtain a fourth covariant point off this tangent plane, thus determining a covariant tetrahedron of reference. This we did in the preceding paper by choosing the second Laplace transform,*

$$\sigma_1 = \sigma_v - \left(b' + \frac{H_v}{H}\right)\sigma = y_{vv} - \left(2b' + \frac{H_v}{H}\right)y_v + b'\left(b' + \frac{H_v}{H} - \frac{b'_v}{b'}\right)y.$$

We propose to substitute for this point another, in the definition of which the two families C_u and C_v, which make up the conjugate net, enter symmetrically. Instead of taking the line $y\sigma_1$ as an edge of the tetrahedron, let us choose the line yz determined by the intersection of the two planes which osculate at the point y the two curves of the net passing through y.

It will simplify our subsequent calculations if we write the first of equations (4) in the form

$$y_{vv} = \alpha y_{uu} + \beta y_u + \gamma y_v + \delta y, \tag{10}$$

where

$$\alpha = \frac{1}{a}, \quad \beta = -\frac{b}{a}, \quad \gamma = -\frac{c}{a}, \quad \delta = -\frac{d}{a}. \tag{11}$$

This is always possible, since $a \neq 0$.

We may write equation (10), or, what is the same thing, the first of equations (4), in the form

$$y_{vv} - \gamma y_v = \alpha y_{uu} + \beta y_u + \delta y.$$

The left-hand member represents a point in the osculating plane to the curve C_v, and the right-hand member a point in the osculating plane to the curve C_u. Consequently, the point

$$z = y_{vv} - \gamma y_v = \alpha y_{uu} + \beta y_u + \delta y \tag{12}$$

lies on the line of intersection of the two osculating planes.

We may therefore associate with each point y of the surface S_y a point z, so that through each point of the surface will pass a definite line yz. With the totality of points y of the surface will be associated ∞^2 lines yz, constituting a congruence which Wilczynski has called the *axis congruence*.† The point z

* Preceding paper, equations (51).

† E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 311-327. The present writer made independent use of the axis congruence, in obtaining the canonical development given below, § 3, without a knowledge of Wilczynski's work. This development was, in fact, announced at the February, 1914, meeting of the American Mathematical Society.

is not, however, a covariant point, and therefore can not serve as a fourth vertex for our tetrahedron of reference. This fourth vertex we may determine as follows: The lines of the axis congruence are tangent (in general) to two surfaces, or focal sheets, so that on each line of the congruence are fixed two points of tangency, or focal points. Either of the focal points might serve as a covariant point; but a unique point may be determined by choosing instead the harmonic conjugate of y with respect to the focal points.

In finding the coordinates of the focal points, we make use of a result given in § 8 of the preceding paper. Let a congruence consist of the lines joining corresponding points of two surfaces S_y and S_z defined by the equations

$$y^{(k)}=f^{(k)}(u,v), \quad z^{(k)}=g^{(k)}(u,v), \qquad (k=1,2,3,4).$$

The functions y and z will satisfy a completely integrable system of partial differential equations of the form

$$\left.\begin{aligned} y_v&=a^{(1)}y+b^{(1)}z+c^{(1)}y_u+d^{(1)}z_v, \\ z_u&=a^{(2)}y+b^{(2)}z+c^{(2)}y_u+d^{(2)}z_v, \\ y_{uu}&=\alpha^{(1)}y+\beta^{(1)}z+\gamma^{(1)}y_u+\delta^{(1)}z_v, \\ z_{vv}&=\alpha^{(2)}y+\beta^{(2)}z+\gamma^{(2)}y_u+\delta^{(2)}z_v, \end{aligned}\right\} \tag{13}$$

where the coefficients in the last two equations have nothing to do with the quantities (6).

The focal sheets will be given by the formulas*

$$z_1=z-\lambda_1 y, \quad z_2=z-\lambda_2 y, \tag{14}$$

where λ_1 and λ_2 are the roots (which may or may not be distinct) of the quadratic

$$d^{(1)}t^2+(c^{(1)}d^{(2)}-d^{(1)}c^{(2)}-1)t+c^{(2)}=0. \tag{15}$$

We need for our purposes only the first two of equations (11), the coefficients of which we now proceed to find. We have

$$z=y_{vv}-\gamma y_v, \tag{12}$$

$$\left.\begin{aligned} z_u&=y_{uvv}-\gamma y_{uv}-\gamma_u y_v \\ &=\alpha^{(3)}y_{vv}+(\beta^{(3)}-b'\gamma)y_u+(\gamma^{(3)}-c'\gamma-\gamma_u)y_v+(\delta^{(3)}-d'\gamma)y, \\ z_v&=y_{vvv}-\gamma y_{vv}-\gamma_v y_v \\ &=(\alpha^{(4)}-\gamma)y_{vv}+\beta^{(4)}y_u+(\gamma^{(4)}-\gamma_v)y_v+\delta^{(4)}y, \end{aligned}\right\} \tag{16}$$

in which use has been made of the last two of equations (5). Replacing y_{vv} in the second of equations (16) by its value $z+\gamma y_v$, and then solving for y_v, the first of equations (13) is obtained. The second of equations (13) may then

---

* Formulas (14) are obviously equivalent to equations (72) of the preceding paper, since all coordinates are homogeneous.

be obtained from the first of equations (16). Recalling that by (6) $\alpha^{(3)}=c'$, we find without difficulty for four of the coefficients in (13):

$$c^{(1)}=-\beta^{(4)}/(\gamma^{(4)}-\gamma_v+\gamma\alpha^{(4)}-\gamma^2), \quad d^{(1)}=1/(\gamma^{(4)}-\gamma_v+\gamma\alpha^{(4)}-\gamma^2),$$
$$c^{(2)}=\beta^{(3)}-b'\gamma+c^{(1)}(\gamma^{(3)}-\gamma_u), \quad d^{(2)}=d^{(1)}(\gamma^{(3)}-\gamma_u).$$

But by (6),

$$\alpha^{(4)}=b'+\gamma-\frac{a_v}{a},$$

so that

$$\gamma^{(4)}-\gamma_v+\gamma\alpha^{(4)}-\gamma^2=\gamma^{(4)}+b'\gamma+\alpha c_v,$$

and the quadratic (15) becomes

$$t^2-(\beta^{(3)}+\gamma^{(4)}+\alpha c_v)t+(\beta^{(3)}-b'\gamma)(\gamma^{(4)}+b'\gamma+\alpha c_v)-\beta^{(4)}(\gamma^{(3)}-\gamma_u)=0. \tag{17}$$

The discriminant of this quadratic is seen without difficulty to be

$$\Delta=(\beta^{(3)}-\gamma^{(4)}-2b'\gamma-\alpha c_v)^2+4\beta^{(4)}(\gamma^{(3)}-\gamma_u); \tag{18}$$

a somewhat lengthy calculation, in which use is made of equations (6), shows that

$$\beta^{(3)}-\gamma^{(4)}-2b'\gamma-\alpha c_v=\frac{1}{a}(d+ab'^2-c'^2+ab'_v-c'_u+b'c+bc')=\frac{\mathfrak{D}}{a}, \tag{19}$$

where $\mathfrak{D}$ is one of the fundamental invariants of the first memoir.* The discriminant of the quadratic (17) is, therefore,

$$\Delta=\frac{\mathfrak{D}^2+4a^2\beta^{(4)}(\gamma^{(3)}-\gamma_u)}{a^2}, \tag{20}$$

and the two roots are

$$\lambda_1, \lambda_2=\frac{\beta^{(3)}+\gamma^{(4)}+\alpha c_v}{2}\pm\frac{\sqrt{\Delta}}{2a}.$$

The two focal sheets of the axis congruence are therefore given by the formulas

$$z_1=z-\lambda_1 y, \quad z_2=z-\lambda_2 y. \tag{14}$$

The expressions just found for the focal points are of course covariants, and either might serve to fix a fourth vertex for our tetrahedron of reference. A more symmetrical choice, and one which leads to a simpler analytic expression, is made by taking as the fourth vertex the harmonic conjugate of y with respect to the two focal points z_1 and z_2. This gives the point

$$\tau=z-\frac{\lambda_1+\lambda_2}{2}y=y_{vv}-\gamma y_v-\frac{1}{2}(\beta^{(3)}+\gamma^{(4)}+\alpha c_v)y. \tag{21}$$

If the invariant Δ given by (20) vanishes, the two focal sheets of the axis congruence coincide with each other and with τ; this can happen only when the

* In the calculation, use is made of equations (24), (29) and (38) of that paper.

lines of the axis congruence are the tangents to a family of asymptotics on a focal sheet.

We shall investigate further the properties of the axis congruence, in its relation to the conjugate net and to associated congruences. First, however, we shall obtain a canonical development for a surface referred to a conjugate net, by making use of the fundamental tetrahedron of reference.

§ 3. *Canonical Development of the Non-homogeneous Coordinates in the Neighborhood of a Point.**

Since the system of differential equations (4) is completely integrable, the derivatives of y to any order are expressible linearly in terms of the fundamental derivatives y_{vv}, y_u, y_v, y. Let us write

$$\frac{\partial^{m+n} y}{\partial u^m \partial v^n} = a^{(m,n)} y_{vv} + b^{(m,n)} y_u + c^{(m,n)} y_v + d^{(m,n)} y. \tag{22}$$

Then we have, in particular, for the coefficients of the first order

$$\begin{array}{llll} a^{(10)}=0, & b^{(10)}=1, & c^{(10)}=0, & d^{(10)}=0, \\ a^{(01)}=0, & b^{(01)}=0, & c^{(01)}=1, & d^{(01)}=0, \end{array}$$

for the coefficients of the second order

$$\begin{array}{llll} a^{(20)}=a, & b^{(20)}=b, & c^{(20)}=c, & d^{(20)}=d, \\ a^{(11)}=0, & b^{(11)}=b', & c^{(11)}=c', & d^{(11)}=d', \\ a^{(02)}=1, & b^{(02)}=0, & c^{(02)}=0, & d^{(02)}=0, \end{array}$$

for the coefficients of the third order the expressions (6), and so on.

Let y be any fixed regular point (on the surface), which we may, without loss of generality, suppose to correspond to the parameter values $u=0$, $v=0$. Then, for any point Y in a sufficiently small neighborhood about the point y, we have the Taylor expansion

$$Y = y + y_u u + y_v v + \frac{1}{2!}(y_{uu} u^2 + 2 y_{uv} uv + y_{vv} v^2) + \ldots\ldots$$

Substituting for the various derivatives of y their expressions in terms of y_{vv}, y_u, y_v, y as given by (22), and rearranging the series, we find the expansion

$$Y = A y_{vv} + B y_u + C y_v + D y, \tag{23}$$

where

$$\left.\begin{aligned} A &= \tfrac{1}{2} a u^2 + \tfrac{1}{2} v^2 + \tfrac{1}{6} a^{(30)} u^3 + \tfrac{1}{2} a^{(21)} u^2 v + \tfrac{1}{2} a^{(12)} u v^2 + \tfrac{1}{6} a^{(03)} v^3 \\ &\quad + \tfrac{1}{24} a^{(40)} u^4 + \tfrac{1}{6} a^{(31)} u^3 v + \tfrac{1}{4} a^{(22)} u^2 v^2 + \tfrac{1}{6} a^{(13)} u v^3 + \tfrac{1}{24} a^{(04)} v^4 + \ldots, \\ B &= u + \tfrac{1}{2} b u^2 + b' uv + \tfrac{1}{6} b^{(30)} u^3 + \tfrac{1}{2} b^{(21)} u^2 v + \tfrac{1}{2} b^{(12)} u v^2 + \tfrac{1}{6} b^{(03)} v^3 + \ldots, \\ C &= v + \tfrac{1}{2} c u^2 + c' uv + \tfrac{1}{6} c^{(30)} u^3 + \tfrac{1}{2} c^{(21)} u^2 v + \tfrac{1}{2} c^{(12)} u v^2 + \tfrac{1}{6} c^{(03)} v^3 + \ldots, \\ D &= 1 + \tfrac{1}{2} d u^2 + d' uv + \ldots, \end{aligned}\right\} \tag{24}$$

where terms omitted are in each case of higher order than those written down.

* The subsequent sections of the paper are independent of the present one.

Let us choose as three vertices of a new tetrahedron of reference the point y and its minus first and first Laplace transform,

$$\rho = y_u - c'y, \quad \sigma = y_v - b'y.$$

A fourth vertex must be linearly dependent on the points y_{vv}, y_u, y_v, y, since these last four are not coplanar. We may therefore take for the fourth vertex any point not in the plane of y, y_u, y_v, given by an expression of the form

$$\tau = y_{vv} - \lambda y_u - \mu y_v - \nu y. \tag{25}$$

We shall seek an expansion having a certain characteristic form, and shall find that the only point τ which yields this expansion is the covariant point τ found in § 2.

Taking τ for the present as defined by (25), we may write the expansion (23) in the form

$$Y = Y_1\tau + Y_2\rho + Y_3\sigma + Y_4 y, \tag{26}$$

where

$$\left.\begin{aligned} &Y_1 = A, \quad Y_2 = B + A\lambda, \quad Y_3 = C + A\mu, \\ &Y_4 = D + A(\nu + c'\lambda + b'\mu) + c'B + b'C. \end{aligned}\right\} \tag{27}$$

The coordinates of the point Y referred to the old tetrahedron of reference are of course

$$Y^{(k)} = Y_1\tau^{(k)} + Y_2\rho^{(k)} + Y_3\sigma^{(k)} + Y_4 y^{(k)}, \qquad (k = 1, 2, 3, 4).$$

Taking now as a new tetrahedron of reference the points τ, ρ, σ, y, we see from this last expression that referred to this new tetrahedron the point Y has the coordinates Y_1, Y_2, Y_3, Y_4, given by equations (27), if the unit-point be properly chosen. We have, therefore, for the coordinates of the point Y referred to the tetrahedron τ, ρ, σ, y the expressions

$$\left.\begin{aligned} Y_1 &= A, \\ Y_2 &= u + \tfrac{1}{2}(b + a\lambda)u^2 + b'uv + \tfrac{1}{2}\lambda v^2 + \tfrac{1}{6}(b^{(30)} + a^{(30)}\lambda)u^3 + \tfrac{1}{2}(b^{(21)} + a^{(21)}\lambda)u^2v \\ &\quad + \tfrac{1}{2}(b^{(12)} + a^{(12)}\lambda)uv^2 + \tfrac{1}{6}(b^{(03)} + a^{(03)}\lambda)v^3 + \dots, \\ Y_3 &= v + \tfrac{1}{2}(c + a\mu)u^2 + c'uv + \tfrac{1}{2}\mu v^2 + \tfrac{1}{6}(c^{(30)} + a^{(30)}\mu)u^3 + \tfrac{1}{2}(c^{(21)} + a^{(21)}\mu)u^2v \\ &\quad + \tfrac{1}{2}(c^{(12)} + a^{(12)}\mu)uv^2 + \tfrac{1}{6}(c^{(03)} + a^{(03)}\mu)v^3 + \dots, \\ Y_4 &= 1 + c'u + b'v + \tfrac{1}{2}[d + a(\nu + c'\lambda + b'\mu) + bc' + b'c]u^2 + (d' + 2b'c')uv \\ &\quad + \tfrac{1}{2}(\nu + c'\lambda + b'\mu)v^2 + \dots, \end{aligned}\right\} \tag{28}$$

in which the terms omitted are of higher order in each case than the ones last written.

We now introduce non-homogeneous coordinates by putting

$$\xi = Y_2/Y_4, \quad \eta = Y_3/Y_4, \quad \zeta = Y_1/Y_4. \tag{29}$$

We shall obtain for ξ, η, ζ power series in u and v, and then by eliminating

from these three series the variables u and v shall express ζ as a power series in ξ and η. We shall ultimately obtain this series up to and including terms of the fourth order; it will, however, simplify our subsequent calculations if we first determine the series up to the third order terms inclusive. We find from the last of equations (28)

$$\frac{1}{Y_4}=1-c'u-b'v+\ldots,$$

so that

$$\left.\begin{aligned}\xi&=Y_2/Y_4=u+\tfrac{1}{2}(b-2c'+a\lambda)u^2+\tfrac{1}{2}\lambda v^2+\ldots,\\ \eta&=Y_3/Y_4=v+\tfrac{1}{2}(c+a\mu)u^2+\tfrac{1}{2}(\mu-2b')v^2+\ldots,\\ \zeta&=Y_1/Y_4=\tfrac{1}{2}au^2+\tfrac{1}{2}v^2+\tfrac{1}{6}(a^{(30)}-3ac')u^3+\tfrac{1}{2}(a^{(21)}-ab')u^2v\\ &\qquad+\tfrac{1}{2}(a^{(12)}-c')uv^2+\tfrac{1}{6}(a^{(03)}-3b')v^3+\ldots.\end{aligned}\right\}\tag{30}$$

From the first two of these we find, correct to terms of the third order inclusive,

$$\begin{aligned}&\xi^2=u^2+(b-2c'+a\lambda)u^3+\lambda uv^2+\ldots, &&\xi^3=u^3+\ldots,\\ &\eta^2=v^2+(c+a\mu)u^2v+(\mu-2b')v^3+\ldots, &&\eta^3=v^3+\ldots,\\ &\xi^2\eta=u^2v+\ldots, \quad \xi\eta^2=uv^2+\ldots.\end{aligned}$$

It is possible, therefore, to express in terms of ξ and η those powers of u and v which occur in the expansion for ζ given by the last of equations (30). We obtain, therefore, the expansion, valid to the terms of the third order,

$$\begin{aligned}\zeta=\tfrac{1}{2}a\xi^2+\tfrac{1}{2}\eta^2+\tfrac{1}{6}(a^{(30)}+3ac'-3ab-3a^2\lambda)\xi^3+\tfrac{1}{2}(a^{(21)}-ab'-c-a\mu)\xi^2\eta\\ +\tfrac{1}{2}(a^{(12)}-c'-a\lambda)\xi\eta^2+\tfrac{1}{6}(a^{(03)}+3b'-3\mu)\eta^3+\ldots.\end{aligned}$$

Remembering that the quantities $a^{(30)}$, $a^{(21)}$, $a^{(12)}$, $a^{(03)}$ are the quantities $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$, $\alpha^{(4)}$ given by equations (6), we may reduce the above expansion to

$$\begin{aligned}\zeta=\tfrac{1}{2}a\xi^2+\tfrac{1}{2}\eta^2+\tfrac{1}{6}(4ac'-2ab+a_u-3a^2\lambda)\xi^3-\tfrac{1}{2}(c+a\mu)\xi^2\eta\\ -\tfrac{1}{2}a\lambda\xi\eta^2+\tfrac{1}{6}(4b'+\gamma-\alpha a_v-3\mu)\eta^3+\ldots.\end{aligned}$$

In the expression for τ given by (25), we have at our disposal the three arbitrary quantities λ, μ, ν. These we shall make determinate by causing certain terms in the expansion for ζ to vanish. The most symmetric way of doing this is to suppose the coefficients of $\xi^2\eta$, $\xi\eta^2$, and $\xi^2\eta^2$ to vanish. Our expansion to terms of the third order shows that in this case we have

$$c+a\mu=0, \quad a\lambda=0.$$

But we have supposed that $a\neq 0$, so that we choose

$$\lambda=0, \quad \mu=-\frac{c}{a}=\gamma.$$

This choice gives us for (25)

$$\tau=y_{vv}-\gamma y_v-\nu y,\tag{31}$$

which is a point on the axis of the point y, that is, on the line of the axis congruence which passes through the point y.

The coefficients of ξ^3 and η^3 become invariants; in fact, it is easy to verify that these are

$$4ac'-2ab+a_u=8a\mathfrak{C}',$$
$$4b'-2\gamma-\alpha a_v=8\mathfrak{B}',$$

where $\mathfrak{B}'$ and $\mathfrak{C}'$ are fundamental invariants of the first memoir, which may be shown to have the above expressions if use be made of equations (24), (29), and (38) of that paper. Consequently, the expansion for ζ, correct to terms of the third order, is

$$\zeta=\tfrac{1}{2}a\xi^2+\tfrac{1}{2}\eta^2+\tfrac{4}{3}a\mathfrak{C}'\xi^3+\tfrac{4}{3}\mathfrak{B}'\eta^3+\ldots\ldots$$

We may now find the terms of the fourth order in this expansion. Putting $\lambda=0$, $\mu=\gamma$ in (28), we find

$$Y_1=A,\quad Y_2=B,$$
$$Y_3=v+c'uv+\tfrac{1}{2}\gamma v^2+\tfrac{1}{6}(c^{(30)}+\gamma a^{(30)})u^3+\tfrac{1}{2}(c^{(21)}+\gamma a^{(21)})u^2v$$
$$+\tfrac{1}{2}(c^{(12)}+\gamma a^{(12)})uv^2+\tfrac{1}{6}(c^{(03)}+\gamma a^{(03)})v^3+\ldots\ldots,$$
$$Y_4=1+c'u+b'v+\tfrac{1}{2}(d+bc'+a\nu)u^2+(d'+2b'c')uv+\tfrac{1}{2}(\nu+b'\gamma)v^2+\ldots\ldots,$$

where A and B are given by (24). We find without difficulty that to terms of the third order

$$\xi=u+\tfrac{1}{2}(b-2c')u^2+\tfrac{1}{6}(b^{(30)}-6bc'+6c'^2-3d-3a\nu)u^3$$
$$+\tfrac{1}{2}(b^{(21)}-2d'-2b'c'-bb')u^2v+\tfrac{1}{2}(b^{(12)}-b'\gamma-\nu)uv^2+\tfrac{1}{6}b^{(03)}v^3+\ldots\ldots,$$
$$\eta=v+\tfrac{1}{2}(\gamma-2b')v^2+\tfrac{1}{6}(c^{(30)}+\gamma a^{(30)})u^3+\tfrac{1}{2}(c^{(21)}+\gamma a^{(21)}-d-bc'-a\nu)u^2v$$
$$+\tfrac{1}{2}(c^{(12)}+\gamma a^{(12)}-c'\gamma-2d'-2b'c')uv^2$$
$$+\tfrac{1}{6}(c^{(03)}+\gamma a^{(03)}-6b'\gamma+6b'^2-3\nu)v^3+\ldots\ldots$$

If, now, we calculated the expression

$$\zeta-\tfrac{1}{2}a\xi^2-\tfrac{1}{2}\eta^2-\tfrac{4}{3}a\mathfrak{C}'\xi^3-\tfrac{4}{3}\mathfrak{B}'\eta^3, \tag{32}$$

we should find that it begins with terms of the fourth order in u and v. We need, therefore, calculate only the terms of the fourth order in the expansions for ζ, ξ^2, η^2, ξ^3, and η^3 in terms of u and v. We shall, however, need the explicit expressions for the terms in u^2v^2 alone. Only the expansions for ζ, ξ^2, and η^2 yield such terms; they are easily found to be:

from ξ^2: $(b^{(12)}-b'\gamma-\nu)u^2v^2$,
from η^2: $(c^{(21)}+\gamma a^{(21)}-d-bc'-a\nu)u^2v^2$,
from ζ : $[\tfrac{1}{4}a^{(22)}-\tfrac{1}{2}c'a^{(12)}-\tfrac{1}{2}b'a^{(21)}$
$\qquad -\tfrac{1}{4}(d+bc'-2c'^2-b'c-2ab'^2+2a\nu)]u^2v^2$.

Multiplying the first of these three by $-a/2$, the second by $-1/2$, and adding

to the last, we obtain the term in u^2v^2 which occurs in the expansion of the quantity (32). The coefficient of this term is, therefore,

$$\begin{aligned}\tfrac{1}{4}a^{(22)}-\tfrac{1}{2}c'a^{(12)}-\tfrac{1}{2}b'a^{(21)}-\tfrac{1}{4}(d+bc'-2c'^2-b'c-2ab'^2+2a\nu)\\ -\tfrac{1}{2}a(b^{(12)}-b'\gamma-\nu)-\tfrac{1}{2}(c^{(21)}+\gamma a^{(21)}-d-bc'-a\nu).\end{aligned} \tag{33}$$

From (6) we find that

$$a^{(12)}=c', \quad a^{(21)}=ab', \quad c^{(21)}=b'c+c'^2+c'_u,$$

since $a^{(12)}=\alpha^{(3)}$, $a^{(21)}=\alpha^{(2)}$, $c^{(21)}=\gamma^{(2)}$. Let us write for $b^{(12)}$ its equivalent $\beta^{(3)}$, without substituting its value from (6). Also, since

$$y_{uvv}=\alpha^{(3)}y_{vv}+\beta^{(3)}y_u+\gamma^{(3)}y_v+\delta^{(3)}y,$$

we have by differentiating with respect to u

$$y_{uuvv}=\alpha^{(3)}y_{uvv}+\alpha^{(3)}_u y_{vv}+\beta^{(3)}y_{uu}+\ldots,$$

which, with the aid of (4) and (5), becomes

$$y_{uuvv}=a^{(22)}y_{vv}+b^{(22)}y_u+c^{(22)}y_v+d^{(22)}y,$$

where, in particular (since $\alpha^{(3)}=c'$),

$$a^{(22)}=c'_u+c'^2+a\beta^{(3)}.$$

We may now easily reduce the expression (33) to

$$\tfrac{1}{4}(d-c'_u-c'^2+bc'-b'c-a\beta^{(3)}+2a\nu),$$

which by (6) may be written

$$\tfrac{1}{4}(2a\nu-a\gamma^{(4)}-a\beta^{(3)}-c_v).$$

Consequently, in the expansion of the expression (32), the coefficient of u^2v^2 may be made to vanish if one chooses

$$\nu=\tfrac{1}{2}(\beta^{(3)}+\gamma^{(4)}+\alpha c_v). \tag{34}$$

Consequently, the fourth vertex of the tetrahedron of reference, which was given by (25) and then reduced to (31), becomes

$$\tau=y_{vv}-\gamma y_v-\tfrac{1}{2}(\beta^{(3)}+\gamma^{(4)}+\alpha c_v)y,$$

which is the covariant (21) found in § 2.

If we note that to terms of the fourth order $u^4=\xi^4$, $u^3v=\xi^3\eta$, $uv^3=\xi\eta^3$ $v^4=\eta^4$, we find that the expansion for ζ is of the form

$$\begin{aligned}\zeta=\tfrac{1}{2}a\xi^2+\tfrac{1}{2}\eta^2+\tfrac{4}{3}a\mathfrak{C}'\xi^3+\tfrac{4}{3}\mathfrak{B}'\eta^3\\ +C^{(40)}\xi^4+C^{(31)}\xi^3\eta+\bigstar+C^{(13)}\xi\eta^3+C^{(04)}\eta^4+\ldots,\end{aligned} \tag{35}$$

in which all of the coefficients are relative invariants of the conjugate net. That these coefficients are indeed invariants follows immediately from the fact that the tetrahedron of reference has covariant points as vertices.

The plane $\zeta=0$ cuts the surface in a curve which has a node at the point $\xi=\eta=\zeta=0$. The tangents to the curve at this node are evidently given by the equation

$$a\xi^2+\eta^2=0, \tag{36}$$

or by the equations

$$\eta\pm\sqrt{-a}\,\xi=0.$$

These latter are of course the equations of the tangents to the asymptotic curves which pass through the point on the surface.

We may obtain from (35) a development in which all of the coefficients are *absolute* invariants. To do this we must remove the one arbitrary feature which still remains in our coordinate system, viz., the unit-point of the system. If we put

$$\xi=\lambda x, \quad \eta=\mu y, \quad \zeta=\nu z,$$

the development (35) becomes

$$\begin{aligned} z=\frac{a\lambda^2}{2\nu}x^2+\frac{\mu^2}{2\nu}y^2+\frac{4a\mathfrak{C}'\lambda^3}{3\nu}x^3+\frac{4\mathfrak{B}'\mu^3}{3\nu}y^3+\frac{\lambda^4}{\nu}C^{(40)}x^4 \\ +\frac{\lambda^3\mu}{\nu}C^{(31)}x^3y+\star+\frac{\lambda\mu^3}{\nu}C^{(13)}xy^3+\frac{\mu^4}{\nu}C^{(04)}y^4+\ldots.. \end{aligned}$$

Let us choose λ, μ and ν so that

$$8a\mathfrak{C}'\lambda^3=\nu, \quad 8\mathfrak{B}'\mu^3=\nu, \quad \nu=\sqrt{a}\,\lambda\mu,$$

then

$$8\sqrt{a}\,\mathfrak{C}'\lambda^2=\mu, \quad 8\mathfrak{B}'\mu^2=\sqrt{a}\,\lambda,$$

and

$$\lambda=\frac{1}{8\sqrt[3]{a^{\frac12}\mathfrak{B}'\mathfrak{C}'^2}}, \quad \mu=\frac{1}{8\sqrt[3]{\alpha^{\frac12}\mathfrak{B}'^2\mathfrak{C}'}}, \quad \nu=\frac{\sqrt{a}}{64\mathfrak{B}'\mathfrak{C}'}, \tag{37}$$

where, as usual, we have written $\alpha=1/a$. The development now takes the final form

$$\begin{aligned} z=\tfrac12(I^{(20)}x^2+I^{(02)}y^3)+\tfrac16(x^3+y^3) \\ +I^{(40)}x^4+I^{(31)}x^3y+\star+I^{(13)}xy^3+I^{(04)}y^4+\ldots, \end{aligned} \tag{38}$$

where all the coefficients are absolute projective invariants, and, in particular,

$$I^{(20)}=\sqrt[3]{\frac{a^{\frac12}\mathfrak{B}'}{\mathfrak{C}'}}, \quad I^{(02)}=\sqrt[3]{\frac{\alpha^{\frac12}\mathfrak{C}'}{\mathfrak{B}'}}.$$

That the said coefficients are indeed *absolute* invariants may be verified as follows: As we have already pointed out, they are certainly invariants, since the tetrahedron of reference is covariant. Now, referring to § 5 of the preceding memoir, we see that if we carry out a transformation of the independent variables,

$$\bar{u}=\phi(u), \quad \bar{v}=\psi(u), \tag{39}$$

on the system of differential equations (4), any invariant $\Theta^{(j,k)}$ of system (4) will be transformed into essentially the same function of the coefficients of the new system of differential equations, and that the transformed function is related to the old by an equation of the form

$$\bar{\Theta}^{(j,k)}=\phi_u^j\psi_v^k\Theta^{(j,k)}.$$

We say that the invariant $\Theta^{(j,k)}$ is of weight (j, k). Now, any coefficient $C^{(j,k)}$ of the expansion (35) is of weight $(j, k-2)$, *i. e.*, under the transformation (39) it becomes

$$\bar{C}^{(j,k)}=\phi_u^j\psi_v^{k-2}C^{(j,k)}. \tag{40}$$

Moreover, the quantities (37) undergo the transformations

$$\bar{\lambda}=\phi_u\lambda, \quad \bar{\mu}=\psi_v\mu, \quad \bar{\nu}=\psi_v^2\nu. \tag{41}$$

Consequently, any coefficient

$$I^{(j,k)}=\frac{\lambda^j\mu^k}{\nu}C^{(j,k)}$$

of the expansion (38) is an absolute invariant, since in virtue of (40) and (41) the invariant $I^{(j,k)}$ is of weight $(0, 0)$, *i. e.*, undergoes the transformation $\bar{I}^{(j,k)}=I^{(j,k)}$.

It remains for us to describe geometrically the choice of the unit-point of our coordinate system which we have just made. Before doing so, however, we shall write out the transformation of coordinates from the tetrahedron y_{vv}, y_u, y_v, y to the tetrahedron which gives rise to the development (38). A point defined by an expression of the form

$$y_1y_{vv}+y_2y_u+y_3y_v+y_4y$$

has y_1, y_2, y_3, y_4 as its coordinates referred to the tetrahedron y_{vv}, y_u, y_v, y. From (27), on putting $\lambda=0$, $\mu=\gamma$, we find for the coordinates of the same point referred to the tetrahedron τ, ρ, σ, y the expressions

$$Y_1=y_1, \quad Y_2=y_2, \quad Y_3=y_3+\gamma y_1, \quad Y_4=y_4+(\nu+b'\gamma)y_1+c'y_2+b'y_3,$$

where

$$\nu=\tfrac{1}{2}(\beta^{(3)}+\gamma^{(4)}+\alpha c_v).$$

Remembering that the coordinates x, y, z of the development (38) are related to these by the equations

$$\lambda x=Y_2/Y_4, \quad \mu y=Y_3/Y_4, \quad \sqrt{a}\lambda\mu z=Y_1/Y_4,$$

we obtain the transformation from the coordinates y_1, y_2, y_3, y_4 referred to

the tetrahedron y_{vv}, y_u, y_v, y, to the coordinates x, y, z of the development (38):

$$\left.\begin{aligned}\lambda x=\frac{y_2}{(\nu+b'\gamma)y_1+c'y_2+b'y_3+y_4},\quad \mu y=\frac{y_3+\gamma y_1}{(\nu+b'\gamma)y_1+c'y_2+b'y_3+y_4},\\ \sqrt{a}\lambda\mu z=\frac{y_1}{(\nu+b'\gamma)y_1+c'y_2+b'y_3+y_4},\end{aligned}\right\}\quad(42)$$

where

$$\lambda=\frac{1}{8\sqrt[3]{a^{\frac{1}{2}}\mathfrak{B}'\mathfrak{C}'^2}},\quad \mu=\frac{1}{8\sqrt[3]{a^{\frac{1}{2}}\mathfrak{B}'^2\mathfrak{C}'}},\quad \nu=\frac{\beta^{(3)}+\gamma^{(4)}+\alpha c_v}{2}.$$

Let x_1, x_2, x_3, x_4 be a set of homogeneous coordinates corresponding to the non-homogeneous coordinates x, y, z. Then we may write

$$\sqrt{a}\lambda\mu x_1=y_1,\quad \lambda x_2=y_2,\quad \mu x_3=y_3+\gamma y_1,\quad x_4=(\nu+b'\gamma)y_1+c'y_2+b'y_3+y_4.\quad(43)$$

We shall need a covariant point, different from the point τ, which does not lie in the tangent plane to the surface. The line joining the first and second Laplace transforms of a point y of the surface is determined by the two points

$$\sigma=y_v-b'y,\quad \sigma_v=y_{vv}-b'y_v-b'_vy,$$

and therefore lies in the osculating plane to the curve C_v at y. The point

$$\sigma_v+(b'-\gamma)\sigma=y_{vv}-\gamma y_v+(b'\gamma-b'^2-b'_v)y=y_{vv}-\gamma y_v+(b'\gamma-\beta^{(3)})y$$

evidently lies on the line $y\tau$. Referred to the tetrahedron y_{vv}, y_u, y_v, y it has coordinates $(1, 0, -\gamma, b'\gamma, -\beta^{(3)})$; using (43), we find that, *referred to the fundamental tetrahedron of reference, the point in which the line joining the first and second Laplace transforms of a point y meets the corresponding axis $y\tau$ has coordinates proportional to*

$$x_1=\frac{64\mathfrak{B}'\mathfrak{C}'}{\sqrt{a}},\ x_2=0,\ x_3=0,\ x_4=\tfrac{1}{2}(\gamma^{(4)}-\beta^{(3)}+\alpha c_v+2b'\gamma)=-\frac{\mathfrak{D}}{2a}.\quad(44)$$

The final expression for x_4 follows from equation (19).

Let us now consider the cubic surface

$$z=\tfrac{1}{2}(I^{(20)}x^2+I^{(02)}y^2)+\tfrac{1}{6}(x^3+y^3),\quad(45)$$

or, in homogeneous coordinates (with reference to the covariant tetrahedron),

$$F\equiv 6x_1x_4^2-3I^{(20)}x_2^2x_4-3I^{(02)}x_3^2x_4-x_2^3-x_3^3=0.$$

Denoting by F_{x_1} the partial derivative $\partial F/\partial x_1$, etc., we find

$$F_{x_1}=6x_4^2,\quad F_{x_2}=-6I^{(20)}x_2x_4-3x_2^2,\quad F_{x_3}=-6I^{(02)}x_3x_4-3x_3^2,$$
$$F_{x_4}=12x_1x_4-3I^{(20)}x_2^2-3I^{(02)}x_3^2.\quad(46)$$

All of these vanish simultaneously if and only if $x_2=x_3=x_4=0$, *i. e.*, the vertex

$(1,0,0,0)$ is the only singular point of the surface. The only one of the second derivatives which does not vanish at this point is $F_{x_4 x_4}$, so that the cone of the second order which is tangent to the cubic surface at the singular point degenerates into the two coincident planes $x_4^2=0$. The surface is, therefore, a *unodal* cubic, the vertex τ $(1, 0, 0, 0)$ of the tetrahedron of reference is its *unode*, and the face $x_4=0$ of the tetrahedron is its *uniplane*.

It is not difficult to see that any unodal cubic surface having as its unode the point $(1,0,0,0)$ and as its uniplane the plane $x_4=0$ must have an equation of the form

$$F \equiv x_1 x_4^2+\phi(x_2, x_3, x_4)=0,$$

where ϕ is a homogeneous polynomial of the third degree in x_2, x_3, x_4. In non-homogeneous coordinates this equation may by written

$$z+\phi(x, y, 1)=0. \tag{47}$$

The arbitrary function ϕ may be determined uniquely by subjecting the cubic surface to a further condition. In fact, let us suppose that the unodal cubic has contact of the third order with our original surface, which is given by the canonical development,

$$z=\tfrac{1}{2}(I^{(20)}x^2+I^{(02)}y^2)+\tfrac{1}{6}(x^3+y^3)+\cdots.$$

Then the value of z obtained from (47) must agree with this expansion to terms of the third order inclusive. Consequently,

$$\phi(x, y, 1)\equiv\tfrac{1}{2}(I^{(20)}x^2+I^{(02)}y^2)+\tfrac{1}{6}(x^3+y^3),$$

which makes (47) coincide with (45). We have then the result:

The cubic surface

$$z=\tfrac{1}{2}(I^{(20)}x^2+I^{(02)}y^2)+\tfrac{1}{6}(x^3+y^3), \tag{45}$$

is completely characterized by the following properties: (1) *It has a unode at the vertex τ of the covariant tetrahedron,* (2) *its uniplane is the face of this tetrahedron opposite the point y, and* (3) *it has contact of the third order with the surface S_y at the point y.**

The tangent plane to the surface S_y at y cuts the unodal cubic in the cubic curve

*The unodal cubic just characterized closely resembles the *canonical cubic* introduced by Wilczynski in his study of surfaces referred to their asymptotic curves. Cf. his "Projective Differential Geometry of Curved Surfaces," second memoir, *Transactions of the American Mathematical Society,* Vol. IX (1908), pp. 104 *et seq.* A unodal cubic may, of course, always be uniquely determined as follows, in relation to a point y of a curved surface. In the tangent plane to the surface at y fix a line l not passing through y, and off the tangent plane fix a point P. Then just one unodal cubic may be determined having contact of the third order with the surface at y, and having P as its unode and the plane passing through P and l as its uniplane.

$$z=0, \quad 3I^{(20)}x^2+3I^{(02)}y^2+2x^3+2y^3=0, \tag{48a}$$

or in homogeneous coordinates

$$x_1=0, \quad 3I^{(20)}x_2^2x_4+3I^{(02)}x_3^2x_4+2x_2^3+2x_3^3=0. \tag{48b}$$

The line $x_1=0$, $x_4=0$, *i. e.*, the edge $\rho\sigma$ of the covariant tetrahedron, cuts this cubic in the three points given by the equation $x_2^3+x_3^3=0$ in conjunction with $x_1=0$, $x_4=0$, *i. e.*, the three points

$$(0, -1, 1, 0), \quad (0, -\varepsilon, 1, 0), \quad (0, -\varepsilon^2, 1, 0), \tag{49}$$

where ε is an imaginary cube root of unity. Consider the point $(0, -1, 1, 0)$. The tangent plane to the unodal cubic surface at this point has the equation

$$x_2+x_3+(I^{(20)}+I^{(02)})x_4=0, \tag{50}$$

and the tangent to the cubic curve (48) at the same point is obtained by adjoining to equation (50) the equation $x_1=0$. Unfortunately, the plane (50) passes through the point τ $(1, 0, 0, 0)$. The point given by equations (44) does not lie in the tangent plane to the surface at y; the plane passing through it and through the tangent to the cubic curve (48) at the point $(0, -1, 1, 0)$ has the equation

$$(I^{(20)}+I^{(02)})(\mathfrak{D}x_1+128\sqrt{a}\mathfrak{B}'\mathfrak{C}'x_4)+128\sqrt{a}\mathfrak{B}'\mathfrak{C}'(x_2+x_3)=0. \tag{51}$$

The unit point of the coordinate system which gives rise to the development (38) *is such that equation* (51), *or, in non-homogeneous coördinates, the equation*

$$(I^{(20)}+I^{(02)})(\mathfrak{D}z+128\sqrt{a}\mathfrak{B}'\mathfrak{C}')+128\sqrt{a}\mathfrak{B}'\mathfrak{C}'(x+y)=0, \tag{52}$$

*represents a plane, which passes through the point of intersection of the line joining the first and second Laplace transforms of the point y with the corresponding axis $y\tau$, and through the line which is tangent to the cubic curve in which the unodal cubic surface is cut by the tangent plane at y, the point of tangency of this line being one of the three points in which the line $\rho\sigma$ intersects the said cubic curve.** *This last fact explains the presence, in the development* (38), *of the cube roots of the invariants*

$$\sqrt{a}\mathfrak{B}'\mathfrak{C}'^2, \quad \sqrt{a}\mathfrak{B}'^2\mathfrak{C}'.$$

We recall that the development (38) was obtained by expanding the nonhomogeneous coordinates ξ, η, ζ of a point of the surface in power series in u and v, and then expressing ζ as a power series in ξ and η by eliminating

*Of course, if $\mathfrak{D}=0$, the point (44) coincides with the point τ, and, therefore, the z coordinate will be absent from equation (52). The unit-point must then be interpreted through the use of some other covariant point off the tangent plane.

between the expansions (30) the parameters u and v. Instead of doing this, we might have eliminated u between the expressions for ξ and η on the one hand, and ξ and ζ on the other, thus obtaining η and ζ as power series

$$\eta = \mathfrak{P}_1(\xi, v), \quad \zeta = \mathfrak{P}_2(\xi, v)$$

in ξ and v. Such expansions would properly be regarded as characteristic of the one-parameter family of curves $C_u(v=\text{const.})$. In the form thus obtained, however, the two power series would not have all of their coefficients invariants —even relative—because the arbitrariness in the choice of the parameter v must still be removed. How this may be done in a case like the present one, so as to yield expansions whose coefficients are all invariants, has been discussed by the author in connection with a one-parameter family of curves in the plane.* The developments for a one-parameter family of space curves were announced by the writer, together with the expansion (38), at a meeting of the American Mathematical Society, to the report of which we refer for the actual forms of the series involved.†

§ 4. *The Parametric Ruled Surfaces and the Developables of the Axis Congruence.*

Let $R^{(u)}$ be the ruled surface generated by the axes yz corresponding to the points of a curve $C_u(v=\text{const.})$ on the surface S_y, and $R^{(v)}$ the ruled surface generated by the axes corresponding to the points of a curve $C_v(u=\text{const.})$ on S_y. The osculating plane to the curve C_u at the point y contains in it the axis of the point y, and also the tangent to the curve C_u. Consequently, this plane is tangent to the ruled surface $R^{(u)}$ at y, since the axis and the tangent to C_u can not coincide. *A parametric curve on the surface S_y is an asymptotic curve on the corresponding ruled surface of the axis congruence.*

It may happen, however, that a parametric ruled surface, say $R^{(u)}$, be developable. Since the generators of a curved developable are its only asymptotics, it follows that the curve C_u can be asymptotic on $R^{(u)}$ only if $R^{(u)}$ be a plane. We examine this point more closely. If the line yz is to generate a developable, then the four points y, z, y_u, z_u must be coplanar. We have

$$\begin{aligned} z &= y_{vv} - \gamma y_v, \\ z_u &= y_{uvv} - \gamma y_{uv} - \gamma_u y_v \\ &= \alpha^{(3)} y_{vv} + (\beta^{(3)} - b'\gamma) y_u + (\gamma^{(3)} - c'\gamma - \gamma_u) y_v + (\delta^{(3)} - d'\gamma) y. \end{aligned}$$

If y, z, y_u, z_u are coplanar, their expressions must be linearly dependent, that is, we must have

* Cf. G. M. Green, "One-parameter Families of Curves in the Plane," *Transactions of the American Mathematical Society*, Vol. XV (1914), pp. 277–290. See § 2.

† *Bulletin of the American Mathematical Society*, Series 2, Vol. XX (1913–14), p. 397.

$$\begin{vmatrix} 1 & -\gamma \\ \alpha^{(3)} & \gamma^{(3)}-c'\gamma-\gamma_u \end{vmatrix}=0.$$

Since by (6) $\alpha^{(3)}=c'$, this reduces to

$$\gamma^{(3)}-\gamma_u=0. \tag{53}$$

This is easily seen to be the condition that the curves C_u be plane curves, as follows: The curves C_u are plane if and only if the points y, y_u, y_{uu}, y_{uuu} be coplanar. But

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \\ y_{uuu} &= ay_{uvv} + a_u y_{vv} + by_{uu} + b_u y_u + cy_{uv} + c_u y_v + dy_u + d_u y \\ &= (a\alpha^{(3)} + ab + a_u) y_{vv} + (\,) y_u + (a\gamma^{(3)} + bc + c'c + c_u) y_v + (\,) y, \end{aligned}$$

in which the coefficients of y_u and y do not concern us. The points y, y_u, y_{uu}, y_{uuu} are thefore linearly dependent if and only if

$$\begin{vmatrix} a & c \\ ac'+ab+a_u & a\gamma^{(3)}+bc+c'c+c_u \end{vmatrix}=0,$$

which, remembering that $\gamma=-c/a$, we may easily reduce to (53). Consequently, *if a parametric ruled surface $R^{(u)}$ of the axis congruence be developable, the corresponding parametric curve C_u is a plane curve and vice versa; the surface $R^{(u)}$ is, in fact, the plane of this curve.*

Unless the invariant (20) vanishes, the lines of the axis congruence may be assembled into two families of developable surfaces. We shall determine these developables presently, but may now state the theorem: *A necessary and sufficient condition that a conjugate net on a surface S_y consist of two families of plane curves, is that the developables of the axis congruence cut the surface S_y in the said conjugate net. The two families of developables will then consist entirely of planes.**

Let us now fix our attention upon a point y of the surface S_y, and the two curves C_u and C_v of the conjugate net which pass through y. Consider the ruled surface $R^{(u)}$, which we suppose non-developable. Any plane through a generator of a skew ruled surface is tangent to that surface at some point of the generator; consequently, the osculating plane to the curve C_v at y, since it passes through the axis at y, must be tangent to $R^{(u)}$ somewhere along that axis. Now, any point on the axis is given by the expression

$$R=z+\nu y,$$

for a suitable value of ν. If we suppose ν to be a function of u, v, then, as the

* The necessity of this condition is of course obvious.

point y traces the curve C_u on S_y, the line yz generates the ruled surface $R^{(u)}$, and the point R traces a curve on $R^{(u)}$. A point on the tangent to this curve at R is found by differentiating the expression for R with respect to u. We have

$$\begin{aligned} R_u &= z_u + \nu y_u + \nu_u y \\ &= c' y_{vv} + (\beta^{(3)} - b'\gamma + \nu) y_u + (\gamma^{(3)} - c'\gamma - \gamma_u) y_v + (\delta^{(3)} - d'\gamma + \nu_u) y. \qquad (54) \end{aligned}$$

We wish to determine the function ν so that for the point y the osculating plane to C_v will be tangent to the surface $R^{(u)}$ at R, and will therefore contain in it the point $R^{(u)}$. Consequently, the points y, y_v, y_{vv}, R_u must be linearly dependent, which by (54) can be the case if and only if

$$\nu = b'\gamma - \beta^{(3)}.$$

The osculating plane to the curve C_v (u=const.) at the point y is tangent to the ruled surface $R^{(u)}$ at the point given by the expression

$$R = y_{vv} - \gamma y_v - (\beta^{(3)} - b'\gamma) y. \qquad (55)$$

Similarly, writing $R' = z + \nu' y$, we have

$$\begin{aligned} R'_v &= z_v + \nu' y_v + \nu'_v y \\ &= (\alpha^{(4)} - \gamma) y_{vv} + \beta^{(4)} y_u + (\gamma^{(4)} - \gamma_v + \nu' y_v) + (\delta^{(4)} + \nu'_v) y, \qquad (56) \end{aligned}$$

which lies in the same plane with y, y_u, and $z = y_{vv} - \gamma y_v$ if and only if

$$\begin{vmatrix} \alpha^{(4)} - \gamma & \gamma^{(4)} - \gamma_v + \nu' \\ 1 & -\gamma \end{vmatrix} = 0,$$

that is, by (6),

$$\nu' = -\gamma\alpha^{(4)} - \gamma^{(4)} + \gamma^2 + \gamma_v = b'\gamma - \dot{\gamma}^{(4)} - \frac{c_v}{a}.$$

The osculating plane to the curve C_u (v=const.) at the point y is tangent to the ruled surface $R^{(v)}$ at the point given by the covariant

$$R' = y_{vv} - \gamma y_v - (\gamma^{(4)} + b'\gamma + \alpha c_v) y. \qquad (57)$$

In § 3 we found an expression which gives the point of intersection of the axis with the line joining the first and second Laplace transforms.* This expression coincides with (55). By symmetry we infer also that (57) gives the point in which the axis is cut by the line joining the minus first and minus second Laplace transforms. It is not difficult to see geometrically why the point R is the intersection of the axis with the line $\sigma\sigma_1$ (σ_1 denoting the first Laplace transform of σ, *i. e.*, the second Laplace transform of y), if we observe that for v=const. the lines $\sigma\sigma_1$ generate a developable surface, which is pre-

---

* See page 300.

cisely the developable enveloped by the osculating planes to the curves $u=$const. at the points where these curves meet the same fixed curve $v=$const.†

The harmonic conjugate of the point y with respect to the points R and R' is

$$\tau=y_{vv}-\gamma y_v-\tfrac{1}{2}(\beta^{(3)}+\gamma^{(4)}+\alpha c_v)y,$$

which is precisely the covariant (21). Consequently, *the harmonic conjugate of the point y with respect to the two focal points of the axis congruence coincides with the harmonic conjugate of y with respect to the two points in which the axis is met by the line joining the first and second Laplace transforms and the line joining the minus first and minus second Laplace transforms.*

The above theorem would be trivial if the points R and R' were the focal points of the axis congruence. Let us determine when this can occur. If the surface S_R, for example, be a focal sheet of the axis congruence, then the axis yR is tangent to the surface S_R. In other words, since the points R_u and R_v lie in the tangent plane to the surface S_R, we must have the four points y, R, R_u, R_v coplanar. Now, putting $\nu=b'\gamma-\beta^{(3)}$ in (54), we find

$$R_u=c'y_{vv}+(\gamma^{(3)}-c'\gamma-\gamma_u)y_v+()y.$$

Also,

$$\begin{aligned}R_v&=y_{vvv}-\gamma y_{vv}-\gamma_v y_v-(\beta^{(3)}-b'\gamma)y_v+()y\\&=(\alpha^{(4)}-\gamma)y_{vv}+\beta^{(4)}y_u+(\gamma^{(4)}-\beta^{(3)}+b'\gamma-\gamma_v)y_v+()y,\\R&=y_{vv}-\gamma y_v-(\beta^{(3)}-b'\gamma)y.\end{aligned}$$

If y, R, R_u, R_v are to be linearly dependent, we must have

$$\begin{vmatrix} c' & 0 & \gamma^{(3)}-c'\gamma-\gamma_u \\ \alpha^{(4)}-\gamma & \beta^{(4)} & \gamma^{(4)}-\beta^{(3)}+b'\gamma-\gamma_v \\ 1 & 0 & -\gamma \end{vmatrix}=0,$$

i. e.,

$$\beta^{(4)}(\gamma^{(3)}-\gamma_u)=0.$$

† Most of the above considerations, and some of those which are to follow, have been extended by the author to the case of general nets—not conjugate—on a curved surface. The axis congruence may be defined for such a net in exactly the same way, as consisting of the lines of intersection of the osculating planes to the curves of the net. The minus first and first Laplace transforms are replaced by the second focal sheets of the two congruences formed by the tangents to the curves C_u and C_v of the net. Calling these covariants ρ and σ, we find that on the surface S_σ the tangent to a curve $u=$const. intersects the axis yz in a point P. But for $v=$const. the tangents to the curves $u=$const. do not form a developable, since the parameter net can not be conjugate on S_σ if it is not on S_y. Nevertheless, the ruled surface $R^{(u)}$ is touched at a point R by the osculating plane to the curve C_v at y, or, as we may say, the ruled surface $R^{(u)}$ is cut along a curve C_R by the developable generated by the planes which osculate the curves C_v at the points where these curves meet a fixed curve C_u. Now, as we have just seen, there is no developable to coincide with this which at the same time takes the place of the one formed by lines joining the first and second Laplace transforms in the case of a conjugate net. We may infer that *the point R is the point of intersection of the axis with the tangent to the curve $u=$const. on S_σ if and only if the parameter net on S_y is conjugate.*

We have already seen that if $\gamma^{(3)}-\gamma_u=0$, the curves C_u are plane. The equation

$$y_{vvv}=\alpha^{(4)}y_{vv}+\beta^{(4)}y_u+\gamma^{(4)}y_v+\delta^{(4)}y$$

shows that if $\beta^{(4)}=0$ the curves C_v are plane. Consequently, *the surface S_R is a focal sheet of the axis congruence if and only if at least one of the families of the conjugate net consists of plane curves. By symmetry, the surface $S_{R'}$ will then be the other focal sheet.*

We now proceed to determine the two families of developables of the axis congruence. Let the point y take the position $y+\delta y$, where, of course, $\delta y=y_u\delta u+y_v\delta v$; then the corresponding point z will move to $z+\delta z$, where $\delta z=z_u\delta u+z_v\delta v$. If y is to move so that the corresponding axis yz generates a developable, the four points y, z, δy, δz must be coplanar. We have

$$\begin{aligned}
\delta y&=y_u\delta u+y_v\delta v,\\
z&=y_{vv}-\gamma y_v,\\
\delta z&=z_u\delta u+z_v\delta v\\
&=[c'\delta u+(\alpha^{(4)}-\gamma)\delta v]y_{vv}+[(\beta^{(3)}-b'\gamma)\delta u+\beta^{(4)}\delta v]y_u\\
&\qquad+[(\gamma^{(3)}-c'\gamma-\gamma_u)\delta u+(\gamma^{(4)}-\gamma_v)\delta v]y_v+(\,)y,
\end{aligned}$$

on putting $v=0$, $v'=0$ in (54) and (56). Let us suppose y, z, δy, δz coplanar, then their expressions must be linearly dependent, and the determinant of the coefficients of y_{vv}, y_u, y_v, y therein must vanish. We thus obtain

$$\begin{vmatrix} 0 & \delta u & \delta v \\ 1 & 0 & -\gamma \\ c'\delta u+(\alpha^{(4)}-\gamma)\delta v & (\beta^{(3)}-b'\gamma)\delta u+\beta^{(4)}\delta v & (\gamma^{(3)}-c'\gamma-\gamma_u)\delta u+(\gamma^{(4)}-\gamma_v)\delta v \end{vmatrix}=0,$$

which is easily reduced to the quadratic

$$a(\gamma^{(3)}-\gamma_u)\delta u^2-\mathfrak{D}\delta u\delta v-a\beta^{(4)}\delta v^2=0, \tag{58}$$

where the invariant $\mathfrak{D}$ is given by (19). This may be regarded as a differential equation defining a net of curves on the surface S_y. Wilczynski has called these the *axis curves.* If a point y of the surface S_y moves along an axis curve, the corresponding axis generates a developable surface of the axis congruence. Through a point y of S_y pass two axis curves; we call the two tangents to these curves at y the *axis tangents* of the point y.

In § 2 we determined the two sheets of the focal surface, and found it necessary to solve the quadratic (17). The form of this quadratic differs from that of (58), though the two have, of course, the same discriminant, given by (20).

If we make use of equations (6), we obtain

$$a\beta^{(4)}=H+2b'_u-b_v, \quad \gamma^{(3)}-\gamma_u=K+2c'_v-\gamma_u,$$

where

$$H=d'+b'c'-b'_u, \quad K=d'+b'c'-c'_v \tag{59}$$

are the Laplace-Darboux invariants of our original conjugate net. We may write (8a) in the form

$$b_v+2c'_v=2b'_u+\gamma_u-\frac{\partial^2}{\partial u\,\partial v}\log a,$$

so that

$$\gamma^{(3)}-\gamma_u=K+2b'_u-b_v-\frac{\partial^2}{\partial u\,\partial v}\log a.$$

We may therefore take the differential equation of the axis curves in the form

$$a\left(K+2b'_u-b_v-\frac{\partial^2}{\partial u\,\partial v}\log a\right)\delta u^2-\mathfrak{D}\delta u\delta v-(H+2b'_u-b_v)\delta v^2=0. \tag{60}$$

The theorems concerning the developables of the axis congruence given at the beginning of this paragraph may of course be read off from (58). We point out one further fact, concerning the relation of the developables of the axis congruence to the parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$. Considering (58) as a quadratic in $\delta v/\delta u$, let us denote its two roots by a_1 and a_2. Then the expression $y_u+a_1 y_v$ represents a point on one of the axis tangents corresponding to the point y. Remembering this, we may without difficulty show that *the plane determined by the axis and either of the axis tangents is tangent to each of the parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ at the two focal points.*

§ 5. *The Ray Congruence, Ray, Anti-Ray and Anti-Axis Curves.*

The quantities

$$\rho=y_u-c'y, \quad \sigma=y_v-b'y \tag{61}$$

are respectively the minus first and first Laplace transforms of y. They both lie in the tangent plane to the surface S_y at y. The surface S_ρ is the second focal sheet of the congruence of tangents to the curves C_u on S_y, and the surface S_σ the second focal sheet of the congruence of tangents to the curves C_v on S_y. Wilczynski has called the line joining the points ρ, σ corresponding to a point y the *ray* of the point y, and the totality of rays, which constitute a congruence, the *ray congruence.* He has also pointed out a dualistic correspondence between the axis and ray congruences.*

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society,* Vol. XVI (1915), pp. 311–327. Cf., in particular, § 3.

In our preceding memoir, we determined the focal sheets of the ray congruence. These are given by the formulas

$$R=\rho+r_1\sigma, \quad S=\rho+r_2\sigma, \tag{62}$$

where r_1 and r_2 are the roots of the quadratic*

$$Hr^2+\mathfrak{D}r-aK=0. \tag{63}$$

We may find the developables of the ray congruence just as we d those of the axis congruence. We determine the curves along which y must move, in order that the corresponding *rays* may generate developables. If y moves to $y+\delta y$, the corresponding ρ and σ move to $\rho+\delta\rho$ and $\sigma+\delta\sigma$, where $\delta\rho=\rho_u\delta u+\rho_v\delta v$, $\delta\sigma=\sigma_u\delta u+\sigma_v\delta v$. We have

$$\begin{aligned}\rho_u&=y_{uu}-c'y_u-c'_uy\\&=ay_{vv}+(b-c')y_u+cy_v+(d-c'_u)y,\\\rho_v&=y_{uv}-c'y_v-c'_vy\\&=b'y_u+(d'-c'_v)y,\\\sigma_u&=c'y_v+(d'-b'_u)y,\\\sigma_v&=y_{vv}-b'y_v-b'_vy,\end{aligned}$$

so that we may write

$$\left.\begin{aligned}&\rho=y_u-c'y, \quad \sigma=y_v-b'y,\\&\delta\rho=a\delta u\cdot y_{vv}+[(b-c')\delta u+b'\delta v]y_u+c\delta u\cdot y_v+[(d-c'_u)\delta u+(d'-c'_v)\delta v]y,\\&\delta\sigma=\delta v\cdot y_{vv}+(c'\delta u-b'\delta v)y_v+[(d'-b'_u)\delta u-b'_v\delta v]y.\end{aligned}\right\} \tag{64}$$

If in its motion the line $\rho\sigma$ is to describe a developable, the four points ρ, σ, $\rho+\delta\rho$, $\sigma+\delta\sigma$ must lie in a plane, *i. e.*, the expressions for ρ, σ, $\delta\rho$, $\delta\sigma$ must be linearly dependent. Equating to zero the determinant of the coefficients of y_{vv}, y_u, y_v, y in (64), and expanding, we obtain as the differential equation which determines the developables of the ray congruence

$$aH\delta u^2-\mathfrak{D}\delta u\delta v-K\delta v^2=0, \tag{65}$$

where $\mathfrak{D}$ is the invariant (19), and H and K are the Laplace-Darboux invariants (59). The differential equation defines a net of curves, the *ray curves*, on the surface S_y. The tangents at y to the two ray curves which pass through y we call the *ray tangents* of the point y.

A net of curves closely related to the ray curves is defined by the differential equation

$$aH\delta u^2+\mathfrak{D}\delta u\delta v-K\delta v^2=0, \tag{66}$$

which differs from that of the ray curves only in the sign of the middle term.

---

* We have changed our notation slightly, so that the quadratic, which follows equations (74) of the preceding paper, differs in form from the present one.

Its roots are therefore the negatives of the roots of (65). Let R_1, R_2 be the ray tangents of the point y, and C_1, C_2 the conjugate tangents, *i. e.*, the tangents at y to the curves of our conjugate parameter net. Let R_1' be the harmonic conjugate of R_1 with respect to C_1 and C_2, and R_2' the harmonic conjugate of R_2 with respect to C_1 and C_2. Then R_1' and R_2' are the tangents at y to the curves of the net defined by (66). This net is therefore defined geometrically in terms of the conjugate net and the net of ray curves. We shall call it the *anti-ray* net, and the tangents at y to the two curves of the net passing through y the *anti-ray tangents. The anti-ray tangents are the harmonic conjugates of the ray tangents with respect to the conjugate tangents.*

In the same way we may define the *anti-axis* curves by means of the differential equation

$$a\left(K+2b_u'-b_v-\frac{\partial^2}{\partial u\partial v}\log a\right)\delta u^2+\mathfrak{D}\,\delta u\,\delta v-(H+2b_u'-b_v)\,\delta v^2=0. \tag{67}$$

The anti-axis tangents are the harmonic conjugates of the axis tangents with respect to the conjugate tangents.

In the preceding paper,* we found that by a transformation of the independent variables,

$$\bar{u}=\phi(u,v),\quad \bar{v}=\psi(u,v)$$

the conjugate net is replaced by a new parameter net, which is asymptotic if and only if ϕ and ψ satisfy the same quadratic partial differential equation of the first order,

$$\phi_u^2+a\phi_v^2=0,\quad \psi_u^2+a\psi_v^2=0.$$

We may throw this into a form analogous to (60) and (65). *The differential equation of the asymptotic curves on the surface S_y is*

$$a\,\delta u^2+\delta v^2=0. \tag{68}$$

Let us now regard the differential equations, of the various nets of curves which we have defined, as binary quadratics in δu, δv. The two roots of any one of the quadratics will give the directions of the two tangents at any point y to the two curves of that particular net which pass through y. The simultaneous invariant of the two quadratics (65) and (68) is $a(H-K)$; if this is zero, the ray tangents separate the asymptotic tangents harmonically. Since $a\neq 0$, we have the theorem of Wilczynski: *A conjugate net has equal Laplace-Darboux invariants if and only if its ray curves form a conjugate net.*

---

* Cf. § 7 thereof.

Taking the differential equation of the axis curves in the form (60), and calculating the simultaneous invariant of this and (68), we find that *the axis curves form a conjugate net if and only if*

$$H-K+\frac{\partial^2}{\partial u\,\partial v}\log a=0.^{*} \tag{69}$$

In the preceding memoir,† we found a necessary and sufficient condition that the congruence of tangents to the curves C_u form a W-congruence. This condition is the vanishing of an invariant which we denoted by W, but the negative of which we now denote by $W^{(u)}$. By making use of equations (24) and (29) of the memoir cited, we find without difficulty that

$$W^{(u)}=2b'_u-b_v-\frac{\partial^2}{\partial u\,\partial v}\log a. \tag{70}$$

The condition was obtained incidentally in relating the formulas of that paper to those of Wilczynski's general theory of congruences. We may, however, obtain it independently, by a procedure similar to that which we now follow in deriving the condition that the congruence of tangents to the family of curves C_v form a W-congruence. This congruence of tangents has the surfaces S_y and S_σ as focal sheets, and will be a W-congruence if and only if the asymptotics on these two surfaces correspond. We have

$$\sigma=y_v-b'y,\quad \sigma_u=c'y_v+(d'-b'_u)y,\quad \sigma_v=y_{vv}-b'y_v-b'_vy,$$
$$\sigma_{uu}=Hy_u+(c'^2+c'_u)y_v+(c'd'+d'_u-b'_{uu})y,$$
$$\sigma_{vv}=(\alpha^{(4)}-b')y_{vv}+\beta^{(4)}y_u+(\gamma^{(4)}-2b'_v)y_v+(\delta^{(4)}-b'_{vv})y,$$

from which five equations we may eliminate y_{vv}, y_u, y_v, y and obtain an equation of the form

$$\sigma_{uu}=a_1\sigma_{vv}+b_1\sigma_u+c_1\sigma_v+d_1\sigma,$$

where, in particular,

$$a_1=H/\beta^{(4)}.$$

This differential equation for σ is of the form of the first of equations (4); consequently, the asymptotics on the surface S_σ are given by the differential equation $a_1\delta u^2+\delta v^2=0$. If the congruence of lines $y\sigma$ is to be a W-congruence,

* The configuration which we are studying is self-dual—concerning which remark see the paper by Wilczynski cited at the beginning of this section. The dualistic correspondence between the axis and ray of a point y is there exhibited in detail. It is therefore to be expected that equation (69) is equivalent to the statement that the Laplace-Darboux invariants of the system of differential equations adjoint to system (4) are equal. This is in fact the case; however, by the adjoint system we do not mean the Lagrange adjoint, but the system of form (4) which is satisfied by the coordinates of the tangent plane to the surface S_y.

† Equations (68).

this differential equation must coincide with equation (68). Therefore, $a_1=a$, so that

$$H-a\beta^{(4)}=0.$$

But from (6), $a\beta^{(4)}=H+2b'_u-b_v$, and we have the result: *A necessary and sufficient condition that the congruence of tangents to the curves C_v on the surface S_y be a W-congruence is the vanishing of the invariant*

$$W^{(v)}=2b'_u-b_v. \tag{71}$$

We may therefore write the differential equation (60) of the axis curves in the form

$$a(K+W^{(u)})\delta u^2-\mathfrak{D}\delta u\delta v-(H+W^{(v)})\delta v^2=0. \tag{72}$$

The method which we have just followed in obtaining the invariant $W^{(v)}$ leads at once to the definition of two important nets on the surface S_y. It is easily verified that ρ satisfies an equation of the form

$$\rho_{uu}=a_{-1}\rho_{vv}+b_{-1}\rho_u+c_{-1}\rho_v+d_{-1}\rho,$$

where

$$a_{-1}=a(\gamma^{(3)}-\gamma_u)/K,$$

so that the asymptotics on the surface S_ρ are given by the differential equation $a_{-1}\delta u^2+\delta v^2=0$. This, however, defines the net of curves on the surface S_y which corresponds to the asymptotic net on S_ρ. *The congruence of tangents to the curves C_u on S_y sets up a point-to-point correspondence between its focal sheets S_y and S_ρ, and the net of curves on S_y which corresponds to the asymptotic net on S_ρ is defined by the differential equation*

$$a(\gamma^{(3)}-\gamma_u)\delta u^2+K\delta v^2=0. \tag{73}$$

Similarly, the differential equation

$$H\delta u^2+\beta^{(4)}\delta v^2=0 \tag{74}$$

defines the net of curves on S_y which corresponds to the asymptotic net on S_σ.

The tangents at a point y to the curves of either of these nets are separated harmonically by the parametric conjugate tangents at y. They coincide with the asymptotic tangents only when the corresponding congruence is a W-congruence.

An interesting special case, to which we shall return later, is that in which the congruences of tangents to the curves C_u and C_v are both of them W-congruences. If, in the differential equation (72) of the axis curves, we put $W^{(u)}=0$, $W^{(v)}=0$, we obtain for this case

$$aK\delta u^2-\mathfrak{D}\delta u\delta v-H\delta v^2=0.$$

But the focal points of the ray corresponding to a point y are given by the formulas

$$R=\rho+r_1\sigma, \quad S=\rho+r_2\sigma, \tag{62}$$

where r_1 and r_2 are the roots of the quadratic

$$Hr^2+\mathfrak{D}r-aK=0. \tag{63}$$

These points are evidently the same as those in which the two axis tangents meet the ray. Moreover, it is easily seen that this can happen only when both $W^{(u)}$ and $W^{(v)}$ vanish. Consequently, *the axis tangents of a point y meet the corresponding ray in the focal points of the ray if and only if both of the congruences of tangents to the curves C_u and C_v on S_y are W-congruences, i. e., $W^{(u)}=0$, $W^{(v)}=0$.*

A glance at equation (65) will show that *the ray tangents meet the ray in the focal points of the ray if and only if the conjugate net on S_y has equal Laplace-Darboux invariants, i. e., $H-K=0$.* This affords a new geometric characterization of a conjugate net with equal Laplace-Darboux invariants.

Many interesting questions present themselves in connection with the various nets which we have defined in this section. Aside from the applications to be made in a later paragraph, we refrain from further consideration of these matters, although an exhaustive study of the interrelations of these nets, and their bearing on the general theory of conjugate nets and congruences, is greatly to be desired.

§ 6. *The Associate Conjugate Net.*

We noted in the preceding section that the asymptotic curves of the surface S_y are given by the differential equation

$$a\delta u^2+\delta v^2=0. \tag{68}$$

We shall now define a new net of curves, which will be useful later. The differential equation

$$a\delta u^2-\delta v^2=0 \tag{75}$$

determines a net of curves, which for reasons which will appear presently we shall call the *associate conjugate net.* The tangents to the two curves of the net at y we shall call the *associate conjugate tangents.* From (68) and (75), we see that the associate conjugate tangents separate the asymptotic tangents harmonically. Consequently, the net defined by (75) is actually a conjugate net. Moreover, the form of equation (75) shows that the associate conjugate tangents separate harmonically the original conjugate tangents. Now, there is evidently one and only one pair of lines which separates harmonically each

of two given pairs. *The associate conjugate net, which is defined by the differential equation*

$$a\delta u^2-\delta v^2, \tag{75}$$

is uniquely characterized by the property, that the tangents to the two curves of the net at a point y are harmonically separated both by the pair of asymptotic tangents and by the pair of original conjugate tangents.

The relation between a conjugate net and its associate conjugate net is of course a reciprocal one. In fact, it may be convenient to speak of two conjugate nets as *associated conjugate nets*, if this relation subsists between them, without distinguishing the nets one from the other.

We shall now set up the completely integrable system of partial differential equations, of form (4), for the surface S_y referred to the associate conjugate net. By the transformation

$$\bar{u}=\phi(u, v), \quad \bar{v}=\psi(u, v), \tag{76}$$

where

$$\phi_u^2-a\phi_v^2=0, \quad \psi_u^2-a\psi_v^2=0, \tag{77}$$

the associate conjugate net is made parametric. We may choose for ϕ and ψ functions satisfying the equations

$$\phi_u+\sqrt{a}\,\phi_v=0, \quad \psi_u-\sqrt{a}\,\psi_v=0. \tag{78}$$

The derivatives of any function $y(u, v)$ undergo the transformations

$$y_u=\bar{y}_u\phi_u+\bar{y}_v\psi_u, \quad y_v=\bar{y}_u\phi_v+\bar{y}_v\psi_v, \tag{79}$$

$$\left.\begin{aligned} y_{uu} &= \bar{y}_{uu}\phi_u^2+2\bar{y}_{uv}\phi_u\psi_u+\bar{y}_{vv}\psi_u^2+\bar{y}_u\phi_{uu}+\bar{y}_v\psi_{uu}, \\ y_{uv} &= \bar{y}_{uu}\phi_u\phi_v+\bar{y}_{uv}(\phi_u\psi_v+\phi_v\psi_u)+\bar{y}_{vv}\psi_u\psi_v+\bar{y}_u\phi_{uv}+\bar{y}_v\psi_{uv}, \\ y_{vv} &= \bar{y}_{uu}\phi_v^2+2\bar{y}_{uv}\phi_v\psi_v+\bar{y}_{vv}\psi_v^2+\bar{y}_u\phi_{vv}+\bar{y}_v\psi_{vv}, \end{aligned}\right\} \tag{80}$$

where $\bar{y}_u$, $\bar{y}_v$, etc., stand for $\partial y/\partial\bar{u}$, $\partial y/\partial\bar{v}$, etc. These formulas hold for any transformation of the form (76). We now impose on ϕ and ψ the conditions (78), from which we have

$$\phi_u=-\sqrt{a}\,\phi_v, \quad \psi_u=\sqrt{a}\,\psi_v, \tag{81}$$

and on differentiation

$$\phi_{uu}=-\sqrt{a}\,\phi_{uv}-\frac{a_u}{2\sqrt{a}}\phi_v, \quad \psi_{uu}=\sqrt{a}\,\psi_{uv}+\frac{a_u}{2\sqrt{a}}\psi_v,$$

$$\phi_{uv}=-\sqrt{a}\,\phi_{vv}-\frac{a_v}{2\sqrt{a}}\phi_v, \quad \psi_{uv}=\sqrt{a}\,\psi_{vv}+\frac{a_v}{2\sqrt{a}}\psi_v.$$

From these we find

$$\left.\begin{aligned}\phi_{uu}&=a\phi_{vv}+\tfrac{1}{2}\left(a_v-\frac{a_u}{\sqrt{a}}\right)\phi_v, \quad \phi_{uv}=-\sqrt{a}\,\phi_{vv}-\frac{a_v}{2\sqrt{a}}\phi_v,\\ \psi_{uu}&=a\psi_{vv}+\tfrac{1}{2}\left(a_v+\frac{a_u}{\sqrt{a}}\right)\psi_v, \quad \psi_{uv}=\sqrt{a}\,\psi_{vv}+\frac{a_v}{2\sqrt{a}}\psi_v.\end{aligned}\right\} \tag{82}$$

In fact, any derivative of ϕ, for instance, is expressible entirely in terms of derivatives of ϕ taken with respect to v alone; the corresponding expression for the same derivative of ψ is then obtained by changing ϕ into ψ and $\sqrt{a}$ into $-\sqrt{a}$. The only derivatives of the third order which we shall need are

$$\left.\begin{aligned}\phi_{uvv}&=-\sqrt{a}\,\phi_{vvv}-\frac{a_v}{\sqrt{a}}\phi_{vv}+\left(\frac{a_v^2}{4a\sqrt{a}}-\frac{a_{vv}}{2\sqrt{a}}\right)\phi_v,\\ \psi_{uvv}&=\sqrt{a}\,\psi_{vvv}+\frac{a_v}{\sqrt{a}}\psi_{vv}+\left(\frac{a_{vv}}{2\sqrt{a}}-\frac{a_v^2}{4a\sqrt{a}}\right)\psi_v.\end{aligned}\right\} \tag{83}$$

Transformation (79) becomes, by (81),

$$y_u=-\sqrt{a}\,\bar{y}_u\phi_v+\sqrt{a}\,\bar{y}_v\psi_v, \quad y_v=\bar{y}_u\phi_v+\bar{y}_v\psi_v, \tag{84}$$

the inverse of which is

$$\bar{y}_u=\frac{1}{2\sqrt{a}\phi_v}(-y_u+\sqrt{a}\,y_v), \quad \bar{y}_v=\frac{1}{2\sqrt{a}\,\psi_v}(y_u+\sqrt{a}\,y_v). \tag{85}$$

Multiplying the last of equations (80) by a, subtracting from the first, and using (78) and (82), we find

$$y_{uu}-ay_{vv}=-4a\bar{y}_{uv}\phi_v\psi_v+\tfrac{1}{2}\left(a_v-\frac{a_u}{\sqrt{a}}\right)\bar{y}_u\phi_v+\tfrac{1}{2}\left(a_v+\frac{a_u}{\sqrt{a}}\right)\bar{y}_v\psi_v, \tag{86}$$

whence from (85)

$$4a\bar{y}_{uv}\phi_v\psi_v=-y_{uu}+ay_{vv}+\frac{a_u}{2a}y_u+\frac{a_v}{2\sqrt{a}}y_v. \tag{87}$$

Also, from the second of (80),

$$y_{uv}=-\sqrt{a}\,\bar{y}_{uu}\phi_v^2+\sqrt{a}\,\bar{y}_{vv}\psi_v^2+\bar{y}_u\phi_{uv}+\bar{y}_v\psi_{uv}; \tag{88}$$

using this, and the equation found by multiplying the last of (80) by a and adding to the first, we obtain without difficulty

$$\left.\begin{aligned}4a\bar{y}_{uu}\phi_v^2&=y_{uu}-2\sqrt{a}\,y_{uv}+ay_{vv}+y_u\left(2\sqrt{a}\frac{\phi_{vv}}{\phi_v}+\frac{a_v}{\sqrt{a}}-\frac{a_u}{2a}\right)-y_v\left(2a\frac{\phi_{vv}}{\phi_v}+\frac{a_v}{2}\right),\\ 4a\bar{y}_{vv}\psi_v^2&=y_{uu}+2\sqrt{a}\,y_{uv}+ay_{vv}-y_u\left(2\sqrt{a}\frac{\psi_{vv}}{\psi_v}+\frac{a_v}{\sqrt{a}}+\frac{a_u}{2a}\right)-y_v\left(2a\frac{\psi_{vv}}{\psi_v}+\frac{a_v}{2}\right).\end{aligned}\right\} \tag{89}$$

The formulas which we have thus far written hold for any function y whatever. We shall now suppose that y is the dependent variable in system (4), and shall obtain the analogous completely integrable system for y as a function of the variables $\bar{u}$, $\bar{v}$. The second of equations (4) is

$$y_{uv}=b'y_u+c'y_v+d'y,$$

so that by (84) we have

$$y_{uv}=\phi_v(-\sqrt{a}\,b'+c')\bar{y}_u+\psi_v(\sqrt{a}\,b'+c')\bar{y}_v+d'\bar{y}, \tag{90}$$

and consequently by substituting in the left-hand member of (88) we may express $\bar{y}_{uu}$ linearly in terms of $\bar{y}_{vv}$, $\bar{y}_u$, $\bar{y}_v$, $\bar{y}$. We shall write the result presently. From the first of equations (4) we have

$$\begin{aligned} y_{uu}-a\,y_{vv}&=b\,y_u+c\,y_v+d\,y \\ &=\phi_v(-\sqrt{a}\,b+c)\bar{y}_u+\psi_v(\sqrt{a}\,b+c)\bar{y}_v+d\bar{y}, \end{aligned}$$

which when substituted in the left-hand member of (86) gives an equation of the Laplace type for $\bar{y}$. We thus find the required system of differential equations.

The completely integrable system of partial differential equations

$$\bar{y}_{uu}=\bar{a}\,\bar{y}_{vv}+\bar{b}\,\bar{y}_u+\bar{c}\,\bar{y}_v+\bar{d}\,\bar{y}, \quad \bar{y}_{uv}=\bar{b}'\,\bar{y}_u+\bar{c}'\,\bar{y}_v+\bar{d}'\,\bar{y}, \tag{91}$$

where

$$\left.\begin{aligned} &\bar{a}=\frac{\psi_v^2}{\phi_v^2}, \quad \bar{b}=\frac{1}{\phi_v}\left(b'-\frac{c'}{\sqrt{a}}-\frac{a_v}{2a}-\frac{\phi_{vv}}{\phi_v}\right), \quad \bar{c}=\frac{\psi_v}{\phi_v^2}\left(\frac{a_v}{2a}-b'-\frac{c'}{\sqrt{a}}+\frac{\psi_{vv}}{\psi_v}\right), \\ &\qquad \bar{d}=-\frac{1}{\phi_v^2}\frac{d'}{\sqrt{a}}, \quad \bar{b}'=\frac{1}{8a\psi_v}\left(a_v-\frac{a_u}{\sqrt{a}}+2\sqrt{a}b-2c\right), \\ &\qquad \bar{c}'=\frac{1}{8a\phi_v}\left(a_v+\frac{a_u}{\sqrt{a}}-2\sqrt{a}\,b-2c\right), \quad \bar{d}'=-\frac{1}{4a\phi_v\psi_v}d, \end{aligned}\right\} \tag{92}$$

has as its integral surfaces the integral surfaces of system (4), *but referred to the associate conjugate net as parameter curves.*

We may define for the associate conjugate net the axis congruence, ray congruence, etc. Let us call these the *associate axis congruence, associate ray congruence,* etc. The associate axis of a point y on the surface is the line joining the point y to the point

$$\bar{z}=\bar{y}_{vv}+\frac{\bar{c}}{\bar{a}}\bar{y}_v.$$

We wish to express this in terms of the coefficients and variables of system (4). Equations (89) hold for any function y; if y satisfy (4), however, we may express $\bar{y}_{vv}$ linearly in terms of y_{vv}, y_u, y_v, y:

$$\begin{aligned} 4a\bar{y}_{vv}\psi_v^2=2ay_{vv}&+\left(b+2\sqrt{a}b'-2\sqrt{a}\frac{\psi_{vv}}{\psi_v}-\frac{a_v}{\sqrt{a}}-\frac{a_u}{2a}\right)y_u \\ &+\left(c+2\sqrt{a}c'-2a\frac{\psi_{vv}}{\psi_v}-\frac{a_v}{2}\right)y_v+(d+2\sqrt{a}d')y. \end{aligned} \tag{93}$$

We find, therefore, using this, (85), and (92),

$$4a\psi_v^2\bar{z}=2ay_{vv}+\left(b-2c'-\frac{a_u}{2a}\right)y_u+\left(c-2ab'+\frac{a_v}{2}\right)y_v+(\,)y,$$

in which the coefficient of y is immaterial. The coefficient of y_u is an invariant; in fact, if use be made of equations (24), (29), and (38) of the previous memoir, we find without difficulty that

$$\mathfrak{C}'=\tfrac{1}{4}\left(2c'-b+\frac{a_u}{2a}\right),\quad \mathfrak{B}'=\tfrac{1}{4}\left(2b'+\frac{c}{a}-\frac{a_v}{2a}\right), \tag{94}$$

consequently,

$$2\psi_v^2\bar{z}=y_{vv}-\frac{2}{a}\mathfrak{C}'y_u+\left(\frac{c}{a}-2\mathfrak{B}'\right)y_v+(\,)y,$$

or

$$2\psi_v^2\bar{z}=z-\frac{2}{a}\mathfrak{C}'y_u-2\mathfrak{B}'y_v+(\,)y. \tag{95}$$

This equation affords a means for interpreting geometrically the invariants $\mathfrak{B}'$ and $\mathfrak{C}'$; this will complete the geometric interpretation of the set of fundamental invariants of the preceding memoir. Remembering that the points z, y, y_u determine the osculating plane to the curve C_u on S_y, and the points z, y, y_v the osculating plane to the curve C_v, we may state the theorem:

The associate axis of a point y of the surface S_y lies in the osculating plane to the curve C_u ($v=const.$) on S_y if and only if the invariant $\mathfrak{B}'$ vanishes; it lies in the osculating plane to the curve C_v ($u=const.$) if and only if the invariant $\mathfrak{C}'$ vanishes.

We recall that if both $\mathfrak{B}'$ and $\mathfrak{C}'$ are identically zero, the surface S_y is a quadric.* Consequently, *the axis congruence of a conjugate net on a surface coincides with the associate axis congruence if and only if the surface is a quadric.*

For the minus first and first Laplace transforms of the associate conjugate net we find without difficulty, using the first of formulas (85),

$$\begin{aligned}8a\phi_v\bar{\rho}&=8a\phi_v(\bar{y}_u-\bar{c}'\bar{y})\\&=4\sqrt{a}(-y_u+\sqrt{a}y_v)-\left(a_v+\frac{a_u}{\sqrt{a}}-2\sqrt{a}b-2c\right)y\\&=4\sqrt{a}(-\rho+\sqrt{a}\sigma)-\left(a_v+\frac{a_u}{\sqrt{a}}-2\sqrt{a}b-2c+4\sqrt{a}c'-4ab'\right)y,\end{aligned}$$

so that

$$2\sqrt{a}\phi_v\bar{\rho}=-\rho+\sqrt{a}\sigma+2(\sqrt{a}\mathfrak{B}'-\mathfrak{C}')y. \tag{96}$$

---

* Preceding paper, end of §7.

Similarly,

$$2\sqrt{a}\psi_v\bar{\sigma}=\rho+\sqrt{a}\sigma+2(\sqrt{a}\mathfrak{B}'+\mathfrak{C}')y. \tag{97}$$

The point of intersection of the ray $\rho\sigma$ of a point y with its associate ray $\bar{\rho}\bar{\sigma}$ is given by

$$(\rho\sigma \# \bar{\rho}\bar{\sigma})=\mathfrak{B}'\rho-\mathfrak{C}'\sigma.$$

If $\mathfrak{B}'=0$, this point coincides with the first Laplace transform σ, and if $\mathfrak{C}'=0$ it coincides with the minus first Laplace transform ρ. Combining these with the results just found in connection with the axis and associated axis, we may state that *if at a point y on the surface S_y the associate axis lies in the osculating plane to the curve C_u (or C_v) then the corresponding ray meets the associate ray in the first Laplace transform (or minus first Laplace transform) of y.*

It is also easily seen that *the quadrics are the only ruled surfaces for which either of the above cases may arise. Moreover, the ray congruence coincides with the associate ray congruence if and only if the axis congruence coincides with the associate axis congruence, in which case the surface S_y is a quadric.*

It is not our purpose here to give an extended discussion of the associate conjugate net in its relation to the original conjugate net. Analytically, a complete discussion would be very complicated, since we should have to express the invariants of the associate net in terms of those of the original net. We may predict the form of some of these expressions. For example, the condition that the surface S_y be ruled is $a\mathfrak{B}'^2+\mathfrak{C}'^2=0$ in terms of the original parameters, and $\bar{a}\bar{\mathfrak{B}}'^2+\bar{\mathfrak{C}}'^2=0$ in terms of the new ones. Consequently, there must be a relation of the form

$$\bar{a}\bar{\mathfrak{B}}'^2+\bar{\mathfrak{C}}'^2=\lambda(a\mathfrak{B}'^2+\mathfrak{C}'^2),$$

in which the factor λ is easily found if use be made of (92) and (82). It should be noted that in this invariant the curves C_u and C_v enter symmetrically; if for an invariant of this kind its expression in the new variables $\bar{u}$, $\bar{v}$ were essentially the same as its expression in the old variables u, v, there would be little interest in studying the associate conjugate net. However, the following example will show that this is not the case.

Let us calculate the invariant $\bar{H}-\bar{K}$. The vanishing thereof is a necessary and sufficient condition that the associate ray curves on S_y form a conjugate net. If we put

$$m=a_v-2c, \quad n=2\sqrt{a}b-\frac{a_u}{\sqrt{a}},$$

we have from (92)

$$\bar{b}'=\frac{1}{8a\psi_v}(m+n),\quad \bar{c}'=\frac{1}{8a\phi_v}(m-n).$$

Consequently, if we denote $\partial\bar{b}'/\partial\bar{u}$ by $\bar{b}'_u$, we have by (85)

$$\bar{b}'_u=\frac{1}{2\sqrt{a}\phi_v}\left(-\frac{\partial\bar{b}'}{\partial u}+\sqrt{a}\,\frac{\partial\bar{b}'}{\partial v}\right),$$

so that

$$16\bar{b}'_u=\frac{1}{\sqrt{a}\phi_v\psi_v}\Bigg[-\frac{1}{a}(m_u+n_u)+\frac{1}{\sqrt{a}}(m_v+n_v)$$
$$+(m+n)\left\{\frac{a_u}{a^2}-\frac{a_v}{a\sqrt{a}}+\frac{\psi_{uv}}{a\psi_v}-\frac{\psi_{vv}}{\sqrt{a}\psi_v}\right\}\Bigg],$$

which by (82) becomes

$$16\bar{b}'_u=\frac{1}{\sqrt{a}\phi_v\psi_v}\left[-\frac{1}{a}(m_u+n_u)+\frac{1}{\sqrt{a}}(m_v+n_v)+(m+n)\left\{\frac{a_u}{a^2}-\frac{a_v}{2a\sqrt{a}}\right\}\right].\quad(98)$$

Similarly,

$$16\bar{c}'_v=\frac{1}{\sqrt{a}\phi_v\psi_v}\left[\frac{1}{a}(m_u-n_u)+\frac{1}{\sqrt{a}}(m_v-n_u)+(m-n)\left\{-\frac{a_u}{a^2}-\frac{a_v}{2a\sqrt{a}}\right\}\right],\quad(99)$$

so that

$$16\sqrt{a}\phi_v\psi_v(\bar{H}-\bar{K})=16\sqrt{a}\phi_v\psi_v(\bar{c}'_v-\bar{b}'_u)$$
$$=\frac{2}{a}m_u-\frac{2}{\sqrt{a}}n_v-\frac{2a_u}{a^2}m+\frac{a_v}{a\sqrt{a}}n.$$

Substituting the values for m and n, we have

$$16\sqrt{a}\phi_v\psi_v(\bar{H}-\bar{K})=\frac{2}{a}(a_{uv}-2c_u)-\frac{2}{\sqrt{a}}\left(2\sqrt{a}b_v+\frac{ba_v}{\sqrt{a}}-\frac{a_{uv}}{\sqrt{a}}+\frac{a_ua_v}{2a\sqrt{a}}\right)$$
$$-\frac{2a_u}{a^2}(a_v-2c)+\frac{a_v}{a\sqrt{a}}\left(2\sqrt{a}b-\frac{a_u}{\sqrt{a}}\right)$$
$$=\frac{4a_{uv}}{a}-\frac{4a_ua_v}{a^2}-\frac{4c_u}{a}-4b_v+\frac{4ca_u}{a^2}.$$

But from the integrability condition (8a), we have

$$b_v+2c'_v=2b'_u-\frac{c_u}{a}+\frac{ca_u}{a^2}-\frac{a_{uv}}{a}+\frac{a_ua_v}{a^2},$$

so that

$$4\sqrt{a}\phi_v\psi_v(\bar{H}-\bar{K})=2c'_v-2b'_u+\frac{2a_{uv}}{a}-\frac{2a_ua_v}{a^2},$$

and finally
$$2\sqrt{a}\phi_v\psi_v(\overline{H}-\overline{K})=c'_v-b'_u+\frac{\partial^2}{\partial u\partial v}\log a$$
$$=H-K+\frac{\partial^2}{\partial u\partial v}\log a. \tag{100}$$

The right-hand member is the invariant appearing in equation (69), the geometric interpretation of which was given in the preceding section. Recalling also Wilczynski's theorem, that for a net with equal Laplace-Darboux invariants the ray curves form a conjugate net, we may state the theorem:

If the axis curves corresponding to a conjugate net themselves form a conjugate net, then the associate ray curves also form a conjugate net, and conversely. If the ray curves form a conjugate net, then the associate axis curves also form a conjugate net, and conversely.

This theorem is sufficient to show that the consideration of the associate conjugate net will not lead to trivial results. Some properties of a conjugate net are enjoyed also by the associate net; we have already seen an obvious instance of this, and in the next section shall find another which is of greater interest, and not at all self-evident. Other properties, however, are not common to the two nets, even when the two component families of each net are concerned in a symmetric way. It would therefore seem desirable *to determine all the properties which hold for both of two associated conjugate nets.* We shall not pursue this study any further, although in the next section we shall make an important application of the associate conjugate net. The subject undoubtedly deserves closer investigation; in fact, it would appear that all properties of a conjugate net might well be described in connection with its associate conjugate net. For instance, the two families of developables of a congruence touch the two focal sheets in a conjugate net on either; if the associate conjugate nets on the two sheets also correspond, the congruence is a W-congruence. The equations of the present paragraph constitute an analytic starting-point for the theory, whose more systematic development we must leave for a future occasion.

§ 7. *Isothermally Conjugate Nets.*

We shall in the present section apply some of the concepts introduced in previous sections to the study of isothermally conjugate nets. If in the second fundamental form of a surface, viz., $D\,\delta u^2+2D'\,\delta u\,\delta v+D''\,\delta v^2$, the coefficient $D'=0$, the parametric net is conjugate, and Bianchi calls it *isothermally conjugate* if in addition $D=D''$, or can be made so by a transformation $\overline{u}=U(u)$, $\overline{v}=V(v)$. This is equivalent to demanding that by such a transformation the

coefficient a in the first of equations (4) be reducible to unity. A necessary and sufficient condition for this is without difficulty seen to be

$$\frac{\partial^2}{\partial u\,\partial v}\log a=0. \tag{101}$$

The conjugate nets on the integral surfaces of system (4) *are isothermally conjugate if and only if*

$$\frac{\partial^2}{\partial u\,\partial v}\log a=0.$$

Isothermally conjugate nets have received increased attention of late. However, until quite recently their only characterization was analytic, until Wilczynski* gave a geometric interpretation of the condition (101). His interpretation consists in the determination of an algebraic relation between three absolute projective invariants which have themselves been previously characterized geometrically. We shall presently give a new geometric interpretation of condition (101) consisting entirely of descriptive geometric properties.

That the left-hand member of (101) is actually a projective invariant of the conjugate net may be seen from equations (70) and (71), from which we have

$$W^{(v)}-W^{(u)}=\frac{\partial^2}{\partial u\,\partial v}\log a. \tag{102}$$

This leads at once to the theorem of Demoulin and Tzitzéica: *If the tangents to the curves C_u and the tangents to the curves C_v on S_y both form W-congruences, the conjugate net C_u, C_v is isothermally conjugate.* In § 5 we gave a geometric characterization of a conjugate net having this property; we found that for such a net the axis tangents at any point of the surface meet the corresponding ray in the focal points of the ray.

We found also in § 5 that the axis curves form a conjugate net if and only if

$$H-K+\frac{\partial^2}{\partial u\,\partial v}\log a=0. \tag{69}$$

Moreover, if $H-K=0$, the ray curves form a conjugate net. Consequently, *if the axis curves and ray curves both form conjugate nets, the parametric conjugate net is isothermally conjugate. An isothermally conjugate net has equal Laplace-Darboux invariants if and only if its axis curves form a conjugate net.*

---

* "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. XVI (July, 1915), pp. 311–327.

A consideration of the differential equations of the axis curves and ray curves shows that *the only isothermally conjugate nets for which the axis curves coincide with the ray curves are those having equal Laplace-Darboux invariants in addition to being subject to the condition of Demoulin and Tzitzéica: the tangents to the curves C_u, and hence also the tangents to the curves C_v, form a W-congruence.*

In § 4 we found that a conjugate net for which both families of curves are plane is completely characterized by the fact that the axis curves coincide with the conjugate net. Since we have

$$a\beta^{(4)}=H+2b'_u-b_v, \quad \gamma^{(3)}-\gamma_u=K+2b'_u-b_v-\frac{\partial^2}{\partial u\,\partial v}\log a,$$

and since $\beta^{(4)}=0$ and $\gamma^{(3)}-\gamma_u=0$ are respectively the conditions that the curves C_v and the curves C_u be plane, *a conjugate net consisting of two families of plane curves is isothermally conjugate if and only if it has equal Laplace-Darboux invariants.*

Surfaces on which one or both families of lines of curvature are plane have been extensively studied; it would seem well worth while to consider also conjugate nets, either general or of some particular kind, for which one or both families are plane. We are not aware of any systematic treatment of the subject, however, in spite of its apparent promise.

Let us return now to the general isothermally conjugate net. In order to give a geometric characterization thereof, we shall use the differential equations of the axis curves, of the anti-ray curves, and of the associate conjugate net. These are, respectively,

$$a\left(K+2b'_u-b_v-\frac{\partial^2}{\partial u\partial v}\log a\right)\delta u^2-\mathfrak{D}\delta u\delta v-(H+2b'_u-b_v)\delta v^2=0, \tag{60}$$

$$aH\delta u^2+\mathfrak{D}\delta u\delta v-K\delta v^2=0, \tag{66}$$

$$a\delta u^2-\quad\delta v^2=0. \tag{75}$$

We recall that the anti-ray tangents are the harmonic conjugates of the ray tangents with respect to the parametric conjugate tangents. Let us regard the above three equations as binary quadratics. The Jacobian of the two binary quadratics

$$a_0x_1^2+2a_1x_1x_2+a_2x_2^2=0, \quad b_0x_1^2+2b_1x_1x_2+b_2x_2^2=0,$$

is

$$(a_0b_1-a_1b_0)x_1^2+(a_0b_2-a_2b_0)x_1x_2+(a_1b_2-a_2b_1)x_2^2=0,$$

and its roots give the pair which separates harmonically each of the pairs

defined by the two quadratics. The Jacobian of the two quadratics (60) and (75) is

$$a\mathfrak{D}\delta u^2+2a\left(H-K+\frac{\partial^2}{\partial u\partial v}\log a\right)\delta u\delta v+\mathfrak{D}\delta v^2=0, \tag{103}$$

and defines the pair of lines which separates harmonically both the pair of axis tangents and the pair of associate conjugate tangents. The Jacobian of the two quadratics (66) and (75) is

$$a\mathfrak{D}\delta u^2+2a(H-K)\delta u\delta v+\mathfrak{D}\delta v^2=0, \tag{104}$$

and defines the pair of lines which separates harmonically both the pair of anti-ray tangents and the pair of associate conjugate tangents.

The two Jacobians (103) and (104) coincide if and only if $\partial^2 \log a/\partial u\partial v=0$, *i. e.*, the parametric conjugate net is isothermally conjugate. This means that the double points of the involution determined by the pair of axis tangents and the pair of associate conjugate tangents coincide in this case with the double points of the involution determined by the pair of anti-ray tangents and the pair of associate conjugate tangents. In other words, the three pairs defined by the quadratics (60), (66), and (75) belong to the same involution. We have then the theorem:

A necessary and sufficient condition that a conjugate net on a surface be isothermally conjugate is that for each point of the surface the pair of axis tangents, the pair of anti-ray tangents, and the pair of associate conjugate tangents form pairs of the same involution.

In the geometric characterization just given, the axis tangents may be replaced by the anti-axis tangents (defined by equation (67)), and the anti-ray tangents at the same time by the ray tangents.

We shall now investigate the nature of the original conjugate net, if its associate conjugate net be isothermally conjugate. We have by (92) $\bar{a}=\psi_v^2/\phi_v^2$, and we must calculate the expression $\partial^2 \log \bar{a}/\partial\bar{u}\partial\bar{v}$ in terms of the coefficients and variables of system (4). This may be done most expeditiously as follows:

$$\begin{aligned}\frac{\partial^2}{\partial\bar{u}\partial\bar{v}}\log\bar{a}&=2\frac{\partial^2}{\partial\bar{u}\partial\bar{v}}\log\psi_v-2\frac{\partial^2}{\partial\bar{u}\partial\bar{v}}\log\phi_v\\&=2\frac{\partial}{\partial\bar{v}}\left(\frac{\partial}{\partial\bar{u}}\log\psi_v\right)-2\frac{\partial}{\partial\bar{u}}\left(\frac{\partial}{\partial\bar{v}}\log\phi_v\right)\\&=2\frac{\partial}{\partial\bar{v}}\left[\frac{1}{2\sqrt{a}\phi_v}\left(-\frac{\partial}{\partial u}\log\psi_v+\sqrt{a}\frac{\partial}{\partial v}\log\psi_v\right)\right]\\&\qquad\qquad-2\frac{\partial}{\partial\bar{u}}\left[\frac{1}{2\sqrt{a}\psi_v}\left(\frac{\partial}{\partial u}\log\phi_v+\sqrt{a}\frac{\partial}{\partial v}\log\phi_v\right)\right],\end{aligned}$$

where we have made use of the formula of differentiation (85). We thus obtain

$$\frac{\partial^2}{\partial u \partial v}\log \bar{a} = \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{a}\phi_v}\left(-\frac{\psi_{uv}}{\psi_v}+\frac{\sqrt{a}\psi_{vv}}{\psi_v}\right)\right] - \frac{\partial}{\partial u}\left[\frac{1}{\sqrt{a}\psi_v}\left(\frac{\phi_{uv}}{\phi_v}+\frac{\sqrt{a}\phi_{vv}}{\phi_v}\right)\right]$$

$$= -\frac{\partial}{\partial v}\left[\frac{1}{\sqrt{a}\phi_v}\cdot\frac{a_v}{2\sqrt{a}}\right] + \frac{\partial}{\partial u}\left[\frac{1}{\sqrt{a}\psi_v}\cdot\frac{a_v}{2\sqrt{a}}\right]$$

$$= -\frac{1}{2}\frac{\partial}{\partial v}\left[\frac{1}{\phi_v}\frac{\partial}{\partial v}\log a\right] + \frac{1}{2}\frac{\partial}{\partial u}\left[\frac{1}{\psi_v}\frac{\partial}{\partial v}\log a\right]$$

$$= -\frac{1}{2\sqrt{a}\phi_v\psi_v}\frac{\partial^2}{\partial u \partial v}\log a. \qquad (105)$$

In the final reduction, use has again been made of (85) and (82).

Equation (105) leads to the important result: *If either of two associated conjugate nets is isothermally conjugate, the other is also.* Isothermal conjugacy is therefore one of the properties of a conjugate net which is also a property of the associate conjugate net. It is in a sense independent of the nature of the surface, since there exist an infinite number of isothermally conjugate nets on any curved surface. We have here, then, a property which is common to both of two associated conjugate nets, or subsists for neither. We have not succeeded in finding any others,* so that it may be interesting to determine just how many of these properties there are. We shall leave this question open for the present.

In the course of this and the preceding memoir, we have studied a single conjugate net, with its related configurations, but have not touched upon the larger questions concerning conjugate nets in general on a surface, nor the properties of conjugate nets which are preserved under certain transformations. We hope soon to give a general theory of the transformation of conjugate nets from a purely projective point of view.

HARVARD UNIVERSITY, *August* 11, 1915.

* We mean, of course, properties which do not depend entirely on the nature of the surface; for instance, if a surface is ruled for one conjugate net thereon, it is ruled for every other net.